

SOME PROPERTIES OF  $\mathbb{C}^*$  IN  $\mathbb{C}^2$ .

MARIUSZ KORAS AND PETER RUSSELL

ABSTRACT. We consider plane curves isomorphic to  $\mathbb{C}^*$ . We prove that with one exception the branches at infinity can be separated by an automorphism of  $\mathbb{C}^2$ . We also give a bound for selfintersection number of the resolution curve.

## 0. INTRODUCTION

**0.1.** Let  $U$  be a closed algebraic curve in  $\mathbb{C}^2$  isomorphic to  $\mathbb{C}^*$ . Let  $\overline{U}$  be the closure of  $U$  in  $\mathbb{P}^2$ . By  $L_\infty$  we denote the line at infinity in  $\mathbb{P}^2$ . Let  $\Phi: \overline{S}' \rightarrow \mathbb{P}^2$  be the resolution of  $\overline{U}$ . By this we mean that  $\Phi^{-1}$  is the minimal sequence of blow ups such that the reduced inverse image of the divisor  $\overline{U} + L_\infty$  is an SNC-divisor. Let  $E'$  be the proper transform of  $\overline{U}$  and let  $D' = \Phi^{-1}(L_\infty)_{red}$ . Let  $L'_\infty$  denotes the proper transform of the line  $L_\infty$  in  $\overline{S}'$ . Let  $\Psi: \overline{S}' \rightarrow \overline{S}$  be the *NC-minimalization of the divisor  $D'$  with respect to  $E'$* , i.e.,  $\Psi$  is the successive contraction of possibly  $L'_\infty$  and then more  $D'$ -components such that  $\Psi(D' + E')$  is a SNC-divisor and each  $(-1)$ -component of  $\Psi(D')$  is a branching component of  $\Psi(D' + E')$ . We put  $D = \Psi(D')$ ,  $E = \Psi(E')$ . Let  $S = \overline{S} \setminus D$ . Of course  $S \simeq \mathbb{C}^2$ . We note that  $E' \cdot D' = E \cdot D = 2$ . Since  $D$  has connected support,  $D + E$  is not a chain.

Embeddings of  $\mathbb{C}^*$  into  $\mathbb{C}^2$  can be divided into two classes. The first class consists of embeddings which admit a *good asymptote*, see 0.2, the second class consists of those without any good asymptote. The embeddings from the first class are completely classified in [C-NKR].

**Definition 0.2.** We say that a rational curve  $L \in \mathbb{P}^2$  is a *good asymptote* of  $U$  if  $L \cap \mathbb{C}^2 \simeq \mathbb{C}^1$ , and  $L$  meets  $\overline{U}$  at most once at finite distance, i.e.,  $L \cdot U \leq 1$ .

Notice that this definition differs slightly from the definition in [C-NKR], but the two definitions are equivalent up to an isomorphism of  $\mathbb{C}^2$ .

The main results of this article are Corollary 2.5 and Theorem 4.16. Corollary 2.5 gives a bound for  $E^2$  and for  $(K_{\overline{S}} + D + E)^2$  in the case where  $U$  does not admit a good asymptote. Theorem 4.16 says that in the case of no good asymptote the branches of  $\overline{U}$  at infinity can be separated by an automorphism of  $\mathbb{C}^2$ . It follows from the classification given in [C-NKR] that with one exception this is also true in case where  $U$  admits a good asymptote. Hence throughout the paper we assume that  $U$  does not have a good asymptote.

Another remarkable property is proved in [Kor].

**Theorem 0.3.**  $\kappa(\overline{S} \setminus E) = -\infty$ .

Theorem 0.3 and a theorem of Coolidge imply that  $U$  can be transformed into a line in  $\mathbb{C}^2$  by a birational automorphism of  $\mathbb{C}^2$ , see [KM].

---

2000 *Mathematics Subject Classification.* Primary: 14R10; Secondary: 14H50.

*Key words and phrases.* Embedding of  $\mathbb{C}^*$ , Cremona transformation, Kodaira dimension.

The first author was supported by Polish Grant N N201 608640. The second author was supported by a grant from NSERC, Canada.

## 1. PRELIMINARIES

In the article we use several notions and results from the theory of open algebraic surfaces. We refer the reader to [M] for any undefined terms here. We will also use some results from T. Fujita's paper [Fu1], particularly § 3.

**1.1.** Let  $\overline{M}$  be a complete, non-singular surface and  $T = \sum_{i=1}^n m_i T_i$  a divisor on  $\overline{M}$  with  $T_1, \dots, T_n$  distinct, irreducible curves.

(i) We write  $\sim$  for linear equivalence of integral divisors. We write  $\equiv$  for numerical equivalence of divisors, both over  $\mathbb{Z}$  and over  $\mathbb{Q}$ .

(ii) We call  $T$  a simple normal crossing divisor (an *SNC*-divisor) if  $T$  is reduced, all its components are smooth and at most two of them meet at any point, and if so, transversally.

(iii) A  $(b)$ -curve on  $\overline{M}$  is a curve  $L \simeq \mathbb{P}^1$  with  $L^2 = b$ .

(iv) An *SNC*-divisor  $T$  is *NC*-minimal if every  $(-1)$ -component of  $T$  is a branching component.

(v) We call  $Q(T) = (T_i \cdot T_j)_{1 \leq i, j \leq n}$  the intersection matrix of a reduced  $T$  and put  $d(T) = \det(-Q(T))$ . We put  $d(T) = 1$  if  $\text{Supp}(T) = \emptyset$ .

(vi) A divisor  $R$  is called *contractible* if it is the minimal resolution divisor of a quotient singular point. Hence  $R$  is a chain composed of smooth rational curves  $R_i$  such that  $R_i^2 \leq -2$  or  $R$  is a fork of smooth rational curves with branches of type  $(2, 2, n)$ ,  $(2, 3, 3)$ ,  $(2, 3, 4)$  or  $(2, 3, 5)$  with and negative branching component.

**1.2.** For the definition of *twig*, *tip*, *bark* of a divisor and their properties we refer to [Fu1, §3] and [M, §2, section 3]. We recall only the definition of a *capacity* of a rational chain. Let

$$R = R_1 + \dots + R_s$$

be a chain of smooth rational curves with dual graph

$$\begin{array}{ccc} \otimes & \text{---} \dots \text{---} & \otimes \\ b_1 & & b_s \end{array}$$

Suppose that  $R$  is admissible, i.e., that  $b_i = R_i^2 \leq -2$ ,  $i = 1, \dots, s$ . We recall that  $Q(R)$  is negative definite and  $d(R) \geq 2$ . We put  $e(R) = \frac{d(R_2 + \dots + R_s)}{d(R)}$ .

**1.2.1.** If  $R_1^2 = -k$ , then  $e(R) \geq \frac{1}{k}$ .

We recall

**1.3.** Let  $T$  be a connected *NC*-minimal divisor consisting of smooth rational curves. Assume  $T$  is not a contractible divisor and let  $T_1, \dots, T_s$  be all the maximal twigs of  $T$ . Then  $\text{Bk}(T)^2 = -\sum e(T_i)$ .

**Lemma 1.4.** There is no curve  $C \subset \overline{S}$  such that  $C \cap S \simeq \mathbb{C}^1$  and  $C \cdot E \leq 1$ .

*Proof.* Let  $L$  be the proper transform of  $C$  in  $\mathbb{P}^2$ . Then  $L$  is a good asymptote of  $\overline{U}$ ; a contradiction.  $\square$

**Corollary 1.5.** If  $E^2 \neq -1$ , then the pair  $(\overline{S}, D + E)$  is relatively minimal (see [M, §2, section 3]) i.e.  $K_{\overline{S}} + D + E \equiv P + \text{Bk}(D + E)$  is the Zariski decomposition, where  $P = (K_{\overline{S}} + D + E)^+$ .

**Lemma 1.6.** (i)  $\overline{\kappa}(\overline{S} \setminus (D + E)) \geq 0$ .

(ii)  $(K_{\overline{S}} + D + E)^2 < 3$ .

*Proof.* (i) If  $\bar{\kappa}(\bar{S} \setminus (D + E)) = -\infty$ , then  $|K_{\bar{S}} + D + E| = \emptyset$ , which implies  $E \cdot D \leq 1$ , but  $E \cdot D = 2$ .  
(ii) By [Mi],  $(K_{\bar{S}} + D + E)^2 \leq 3(\chi(S \setminus E) + \frac{1}{12}N^2)$ , where  $N = (K_{\bar{S}} + D + E)^-$  is the negative part in the Zariski decomposition of the divisor  $K_{\bar{S}} + D + E$ . If  $N \neq 0$  we are done since  $\chi(S \setminus E) = 1$ . Suppose that  $N = 0$ . It follows that the divisor  $D + E$  has no twigs (see [Fu1, §3], [M, §2, section 3]). Hence  $D$  is a chain and  $E$  meets the tips of  $D$ . If  $D$  has only one component, then  $D^2 = 1$ . Hence in all cases no component of  $D$  is a  $(-1)$ -curve since  $D$  is NC-minimal w.r.t  $E$ , see 0.1. Clearly  $D$  is not an admissible chain, see 1.2. Therefore there exists a component  $D_1$  of  $D$  such that  $D_1^2 \geq 0$ . By some blowing up and down within  $D$  we can transform  $D$  into a chain  $\Delta$  with a tip  $\Delta_1$  such that  $\Delta_1^2 = 0$  and  $E \cdot \Delta_1 = 1$ . The linear system  $|\Delta_1|$  induces a  $\mathbb{C}$ -ruling of  $S$  with  $E$  as a 1-section. The proper transform in  $\mathbb{P}^2$  of a general member of the system is a good asymptote of  $U$ , contrary to our assumption.  $\square$

**1.7.** Write

$$(K_{\bar{S}} + D + E)^2 = 2 - \varepsilon$$

where  $\varepsilon \geq 0$ . We have  $K_{\bar{S}} \cdot (K_{\bar{S}} + D + E) = 2 - \varepsilon$ . We put  $\gamma = -E^2$ .

**Lemma 1.8.**  $\gamma > 0$ .

*Proof.* Suppose that  $\gamma \leq 0$ . After blowing up over one of the points in  $E \cap D$  we may assume that  $E^2 = 0$ . Therefore  $U$  is a fiber of a  $\mathbb{C}^*$ -ruling of  $\mathbb{C}^2$ . There is a singular fiber with an irreducible component isomorphic to  $\mathbb{C}$ . This is a good asymptote of  $U$  and we reach a contradiction.  $\square$

**1.9.** Let  $R_1, \dots, R_s$  be all maximal twigs of  $D + E$ . Let  $e_i = e(R_i)$ .

**Lemma 1.10.**  $-(Bk(D + E))^2 = \sum e_i \leq 1 + \varepsilon$ .

*Proof.* Suppose that  $\gamma \neq 1$ . Then, by 1.5, the pair  $(\bar{S}, D + E)$  is minimal. Let  $K_{\bar{S}} + D + E = P + Bk(D + E)$  be the Zariski decomposition. By Langer's version [L] of the Kobayashi inequality, see 1.13,  $0 \leq P^2 \leq 3\chi(S \setminus E) = 3$ . We have  $P^2 = (K_{\bar{S}} + D + E)^2 - (Bk(D + E))^2 = 2 - \varepsilon + \sum e_i$ , so we are done. Suppose that  $E^2 = -1$ . We pass to a minimal model of the pair  $(\bar{S}, D + E)$ . In view of 1.4, we possibly contract  $E$  and further components of  $D + E$ , but we do not touch any of the maximal twigs of  $D + E$ . To the resulting divisor we apply the Kobayashi inequality and get the result.  $\square$

**Lemma 1.11.**  $\varepsilon \leq 3$ .

*Proof.* We claim that  $\bar{S}$  is not isomorphic to a relatively minimal rational surface. By 1.8,  $E^2 < 0$ . Hence  $\bar{S}$  is not isomorphic to  $\mathbb{P}^2$ , and if it is isomorphic to a Hirzebruch surface, then  $E$  is the only negative curve in  $\bar{S}$ . Also  $D = D_1 + D_2$  has two irreducible components since the irreducible components of  $D$  generate  $\text{Pic}(\bar{S})$  freely. Now  $D_1^2 \geq 0, D_2^2 \geq 0$ , and since  $E \cdot (D_1 + D_2) = 2$  we may assume  $E \cdot D_1 \leq 1$ , say. After some blowing up we may assume that  $D_1^2 = 0$  and then the proper transform of a general member of the system  $|D_1|$  is a good asymptote of  $U$ , in contradiction to our assumption.

Suppose that  $\varepsilon \geq 4$ . We have  $(K_{\bar{S}} + D) \cdot (K_{\bar{S}} + E) = 2 - \varepsilon - K_{\bar{S}} \cdot E + (K_{\bar{S}} + D) \cdot E = 4 - \varepsilon \leq 0$ . Since  $E \cdot D = 2$ ,  $|E + K_{\bar{S}} + D| \neq \emptyset$ . By Fujita's Theorem [Fu2] there exists  $m$  such that  $|E + m(K_{\bar{S}} + D)| \neq \emptyset$ , but  $|E + (m + 1)(K_{\bar{S}} + D)| = \emptyset$ . We write

$$E + m(K_{\bar{S}} + D) = \sum A_i$$

with each  $A_i$  reduced and irreducible. We have  $|A_i + D + K_{\bar{S}}| = \emptyset$  for every  $i$ . By a standard argument, [Ru2, 2.1, 2.2] for example,  $A_i$  is a smooth rational curve and  $A_i \cdot D \leq 1$ . This implies  $A_i \neq E$ . Since  $\bar{S}$  is not a relatively minimal surface we may assume that  $A_i^2 < 0$  for every  $i$ . (If  $A_i^2 \geq 0$  we replace  $A_i$  by a suitable singular member of the linear system  $|A_i|$ ). We obtain  $-2 \geq E \cdot (K_{\bar{S}} + E) + m(K_{\bar{S}} + D) \cdot (K_{\bar{S}} + E) = \sum A_i \cdot (K_{\bar{S}} + E)$ . Hence there exists  $A_1$  such that  $A_1 \cdot (K_{\bar{S}} + E) < 0$ . It follows that  $A_1 \cdot K_{\bar{S}} < 0$ . Hence  $A_1^2 = -1$  and  $A_1 \cdot E = 0$ . Since  $A_1 \cdot D \leq 1$ ,  $A_1$  is not a branching component of  $D$ . Since  $A_1 \cdot E = 0$  it follows from the NC-minimality of  $D$  w.r.t.  $E$  (see 0.1) that  $A_1$  is not a component of  $D$ . Now the proper transform of  $A_1$  in  $\mathbb{P}^2$  is a good asymptote of  $U$ , a contradiction.  $\square$

**Lemma 1.12.** *Let  $\overline{M}$  be a smooth projective surface. Let  $r$  be the rank the Neron-Severi group  $NS(\overline{M})$ . Then for any set  $C_1, \dots, C_r$  of distinct irreducible curves in  $\overline{M}$  the matrix  $[C_i \cdot C_j]$  is not negative definite.*

*Proof.* Suppose it is. Then in particular  $C_1, \dots, C_r$  are independent in  $NS(\overline{M}) \otimes \mathbb{Q}$  and hence form a basis of  $NS(\overline{M}) \otimes \mathbb{Q}$ . We reach a contradiction with the Hodge Index Theorem.  $\square$

We will use an inequality of Bogomolov-Miyaoka-Yau type (simply the BMY-inequality) proved by R. Kobayashi, S. Nakamura and F. Sakai, [GM, Lemma 8 and Corollary 9] (see also [Mi, Chapter 2, Theorem 6.6.2]). We state it as follows.

**Lemma 1.13.** *Let  $\overline{X}$  be a smooth projective surface and let  $D$  be an SNC-divisor on  $\overline{X}$ . Let  $D_1, \dots, D_k$  be the connected components of  $D$  which are contractible divisors, see 1.1(vi). Let  $G_i, 1 \leq i \leq k$ , be the local fundamental group at the singular point obtained by the contraction of  $G_i$  to point. Assume that the pair  $(X, D)$  is almost minimal. Suppose that  $\overline{\kappa}(X \setminus D) \geq 0$ . Then*

$$((K_{\overline{X}} + D)^+)^2 \leq 3(\chi(X \setminus D) + \sum_{i=1}^k \frac{1}{|G_i|}),$$

The original BMY-inequality was proved in case  $\overline{\kappa}(X \setminus D) = 2$ . A. Langer [L] has extended it to the case  $\overline{\kappa}(X \setminus D) = 0, 1$ .

## 2. BASIC INEQUALITY

**2.1.** Let  $\psi: \overline{S} \rightarrow \overline{N}$  be a 2-reduction of the divisor  $D$  with respect to  $E$ , i.e.,  $\psi$  is a sequence of successive contractions of  $(-1)$ -curves in  $D$  meeting  $E$  (and its successive images) once and such that the divisors  $T = \psi(D)$  and  $E_0 = \psi(E)$  satisfy the following:

- (i)  $T$  is an NC-divisor.
- (ii) for any  $(-1)$ -component  $T_i$  of  $T$ ,  $T_i \cdot E_0 \geq 2$  or  $T_i$  is a branching component of  $T$ .

Note that only curves meeting  $E$  once are contracted. In particular,  $E_0$  is smooth and hence  $E_0 \simeq \mathbb{P}^1$ .

**2.2.** Let  $t$  denote the number of sprouting contractions in  $\psi$ . A subdivisinal blowing down does not change the quantities  $K \cdot (K + D)$  and  $E \cdot (K + D)$ . Under a sprouting blowing down  $K \cdot (K + D)$  increases by 1 and  $E \cdot (K + D)$  decreases by 1. Here, by abuse of notation,  $K$  denotes the canonical divisor of the image of  $\overline{S}$  at some stage of the contraction process  $\psi$ , and the images of  $E$  and  $D$  are denoted by the same letters. Hence

- (a)  $(E_0 + K_{\overline{N}}) \cdot (K_{\overline{N}} + T) = (E + K_{\overline{S}}) \cdot (K_{\overline{S}} + D)$ .
- (b)  $(E_0 + 2K_{\overline{N}}) \cdot (K_{\overline{N}} + T) = (E + 2K_{\overline{S}}) \cdot (K_{\overline{S}} + D) + t = 6 - 2\varepsilon - K_{\overline{S}} \cdot E + t = 8 - 2\varepsilon - \gamma + t$ .

We note the following for future reference.

**2.2.1** A contribution (of 1) to  $t$  arises when there is a  $(-1)$ -curve in  $D$  that is non-branching in  $D$ , meets  $E$  once and has attached to it a maximal twig  $T$  of  $D + E$  consisting of  $(-2)$ -curves. Note that if  $\tau$  their number, then  $T$  contributes  $\frac{\tau}{\tau+1}$  to  $\sum e_i$  in 1.10.

**Proposition 2.3.** *Suppose that  $(E_0 + 2K_{\overline{N}})(K_{\overline{N}} + T) \leq 0$  and that  $\overline{N}$  is not a Hirzebruch surface or  $\mathbb{P}^2$ . Then there exists a  $(-1)$ -curve  $A$  in  $\overline{N}$  such that  $A \cdot E_0 \leq 1$ .*

*Proof.* Suppose that such a curve does not exist. Let  $C_1, C_2$  be the components of  $D$  which meet  $E$ . It may happen that  $C_1 = C_2$ .

**Sub-Lemma 2.3.1.** *There is no curve  $B$  in  $\overline{N}$  such that  $(E_0 + 2K_{\overline{N}}) \cdot B < 0$ .*

*Proof.* Suppose  $B$  exists. Suppose first that  $|B + K_{\overline{N}} + T| \neq \emptyset$ . Let  $F_m = B + m(K_{\overline{N}} + T)$ . Arguing as in the proof of Lemma 1.11, we find  $m$  such that  $|F_m| \neq \emptyset$  and  $|F_{m+1}| = \emptyset$  and we have  $B + m(K_{\overline{N}} + T) = \sum B_i$ . Then  $|B_i + K_{\overline{N}} + T| = \emptyset$  for every  $i$ . By the assumption in Proposition 2.3 we have  $0 > B \cdot (E_0 + 2K_{\overline{N}}) \geq \sum B_i \cdot (E_0 + 2K_{\overline{N}})$ . Hence there exists  $B_i$  such

that  $B_i \cdot (E_0 + 2K_{\overline{N}}) < 0$ .

Free to replace  $B$  by  $B_i$ , we may assume that  $|B + K_{\overline{N}} + T| = \emptyset$ . Then  $B$  is a smooth rational curve and  $B \cdot T \leq 1$ . In particular  $B \neq E_0$  since  $E_0 \cdot T \geq 2$ . So  $B \cdot E_0 \geq 0$  and  $K_{\overline{N}} \cdot B < 0$ , i.e.,  $B^2 \geq -1$ . Suppose that  $B^2 \geq 0$ . Since  $\overline{N}$  is not a minimal rational surface there exists a singular member  $\sum B_j$  of  $|B|$  such that  $B_j^2 < 0$  for every  $j$ . There exists  $B_j$  such that  $B_j \cdot (E_0 + 2K_{\overline{N}}) < 0$ . It follows that  $B_j \cdot K_{\overline{N}} < 0$  hence  $B_j^2 = -1$ , and of course  $|B_j + K_{\overline{N}} + T| = \emptyset$ . We may replace  $B$  by  $B_j$ . Then  $K_{\overline{N}} \cdot B = -1$ , which implies  $B \cdot E_0 \leq 1$ , and  $B$  gives a good asymptote for  $U$ , a contradiction. The sub-lemma is proved.  $\square$

By Theorem 0.1 we have  $\kappa(K_{\overline{N}} + E_0) = -\infty$ . We argue as in [KM, theorem 2.1].

(i) Suppose that  $K_{\overline{N}} \cdot (K_{\overline{N}} + E_0) \leq 0$ . Let  $L$  be a  $(-1)$ -curve in  $\overline{N}$ . Since  $L \cdot E_0 \geq 2$ ,  $|L + K_{\overline{N}} + E_0| \neq \emptyset$ . As above we find  $m \geq 1$  such that we have

$$F = L + m(K_{\overline{N}} + E_0) = \sum A_i$$

with, for each  $i$ ,  $A_i \simeq \mathbb{P}^1$ ,  $A_i \cdot E_0 \leq 1$  and  $A_i^2 < 0$ . Since  $F \cdot K_{\overline{N}} < 0$ , there exists  $A_j$  such that  $A_j \cdot K_{\overline{N}} < 0$ . Hence  $A_j^2 = -1$ , so  $A_j \cdot (E_0 + 2K_{\overline{N}}) < 0$ , and we get contradiction with Lemma 2.3.1.

(ii) Suppose that  $K_{\overline{N}} \cdot (K_{\overline{N}} + E_0) \geq 1$ . Then  $-K_{\overline{N}} - E_0 \geq 0$  by the Riemann-Roch theorem and, in fact,  $-K_{\overline{N}} - E_0 > 0$  since  $E_0 \cdot (-K_{\overline{N}} - E_0) = 2$ . Let again  $L$  be a  $(-1)$ -curve in  $\overline{N}$ . Write  $L = L + K_{\overline{N}} + E_0 + (-K_{\overline{N}} - E_0)$ . Since  $h^0(L) = 1$  and  $L + K_{\overline{N}} + E_0 \geq 0$ ,  $L + K_{\overline{N}} + E_0 = 0$ . There exists a component  $T_i = \psi(D_i)$  of  $T$  such that  $T_i \cdot L > 0$ . Then  $T_i \cdot (K_{\overline{N}} + E_0) = T_i \cdot (-L) < 0$ . It follows that  $K_{\overline{N}} \cdot T_i < 0$ . We obtain  $T_i \cdot (E_0 + 2K_{\overline{N}}) = T_i \cdot (E_0 + K_{\overline{N}}) + T_i \cdot K_{\overline{N}} < 0$  in contradiction to lemma 2.3.1.  $\square$

**Proposition 2.4.**  $(E_0 + 2K_{\overline{N}})(K_{\overline{N}} + T) > 0$

*Proof.* We keep notation of 2.1 and 2.2. We have  $\kappa(K_{\overline{N}} + E_0) = -\infty$ . Suppose that  $(E_0 + 2K_{\overline{N}}) \cdot (K_{\overline{N}} + T) \leq 0$ . Suppose first that  $\overline{N}$  is not isomorphic to a Hirzebruch surface or  $\mathbb{P}^2$ . Let  $A$  be a curve as in Proposition 2.3. Suppose that  $|A + K_{\overline{N}} + T| \neq \emptyset$ . We again find  $m$  such that  $|A + m(K_{\overline{N}} + T)| \neq \emptyset$  and  $|A + n(K_{\overline{N}} + T)| = \emptyset$  for  $n > m$  and we write

$$F = L + m(K_{\overline{N}} + T) = \sum A_i$$

with, for each  $i$ ,  $A_i \simeq \mathbb{P}^1$ ,  $A_i \cdot T \leq 1$  and  $A_i^2 < 0$ .

We have  $0 > (E_0 + 2K_{\overline{N}}) \cdot (A + m(K_{\overline{N}} + T)) = \sum (E_0 + 2K_{\overline{N}}) \cdot A_i$ . Thus there exists  $A_j$  such that  $A_j \cdot (E_0 + 2K_{\overline{N}}) < 0$ . Since  $A_0 \cdot T \leq 1$ ,  $A_j \neq E_0$ . Thus  $A_j \cdot K_{\overline{N}} < 0$ . Since  $A_j^2 < 0$  we obtain  $A_j^2 = -1$ . It follows that  $E_0 \cdot A_j \leq 1$ . Therefore we may assume that  $|A + K_{\overline{N}} + T| = \emptyset$ . Then  $A \cdot T \leq 1$ , which implies that  $A$  is not a branching component of  $T$ . Also  $A \cdot E_0 \leq 1$ . By the properties of  $T$  it follows that  $A$  is not a component of  $T + E_0$ . The proper transform of  $A$  in  $\mathbb{P}^2$  is a good asymptote of  $U$  and we reach a contradiction.

We have already seen that  $\overline{N}$  cannot be isomorphic to  $\mathbb{P}^2$ . Suppose then that  $\overline{N}$  is isomorphic to a Hirzebruch surface. Since the irreducible components of  $T$  generate  $\text{Pic}(\overline{N})$  freely,  $T$  has exactly two components. Write  $T = T_1 + T_2$ . Computing the determinant of  $T$  we get  $-1 = T_1^2 T_2^2 - 1$ . We may assume therefore that  $T_1^2 = 0$ . Let  $T_2^2 = -n$ . Let  $a = T_1 \cdot E_0$ ,  $b = T_2 \cdot E_0$ . Then

$$E_0 \sim (an + b)T_1 + aT_2.$$

Now  $p_a(E_0) = 0$  implies  $-2 = (an + b)(2a - 2) + a(n - 2 - an)$  and consequently  $(a - 1)(an + 2b - 2) = 0$ . Thus

(i)  $a \leq 1$

or

(ii)  $a \geq 2$  and  $2 = an + 2b$ .

In case (i) the proper transform in  $\mathbb{P}^2$  of a general member of the system  $|T_1|$  is a good asymptote of  $U$ , so (i) cannot occur.

Consider (ii). We have  $E_0^2 = a^2n + 2ab = a(an + 2b) = 2a$ . Suppose that  $b = 0$ . Then  $2 = an$ , so  $a = 2$  and  $n = 1$ . But then  $T_2^2 = -1$  and  $T = T_1 + T_2$  is not 2-reduced w.r.t.  $E$ . Hence  $b > 0$ . Suppose that  $b = 1$ . Then  $an = 0$ , hence  $n = 0$ , but then the proper transform in  $\mathbb{P}^2$  of a general member of the system  $|T_2|$  is a good asymptote of  $U$ . Thus  $b \geq 2$ . Let  $q = T_1 \cap T_2$ .

Suppose that  $q \notin E_0$ . Then  $E_0$  intersects  $T_1$  and  $T_2$  in points  $p_1$  and  $p_2$  respectively. The inverse of  $\psi$  involves blowing up over  $p_1$   $a$  times and blowing up over  $p_2$   $b$  times. We have  $t = 2$  (i.e.  $\psi$  involves two sprouting contractions w.r.t.  $D$ ) and  $K_{\overline{N}} \cdot (K_{\overline{N}} + T) = 2 + K_{\overline{S}} \cdot (K_{\overline{S}} + D) = 4 - \varepsilon - K_{\overline{S}} \cdot E$  by 2.2 (b).  $\psi$  involves  $a + b$  contractions on  $E$ , hence  $K_{\overline{N}} \cdot E_0 = K_{\overline{S}} \cdot E - (a + b)$ . We obtain that  $K_{\overline{N}} \cdot (K_{\overline{N}} + T) = 4 - \varepsilon - K_{\overline{N}} \cdot E_0 - a - b = 6 - \varepsilon + a - b$ . On the other hand  $K_{\overline{N}} \cdot (K_{\overline{N}} + T) = 8 + K_{\overline{N}} \cdot T_1 + K_{\overline{N}} \cdot T_2 = 4 + n$ . Hence

$$(*) \quad n = 2 - \varepsilon + a - b$$

By our assumption

$$(**) \quad (E_0 + 2K_{\overline{N}}) \cdot (K_{\overline{N}} + T) = 6 - a + b + 2n = 8 - \varepsilon + n \leq 0.$$

From (ii), since  $b = 2 - \varepsilon + a - n$ , we get

$$(***) \quad (a - 2)(n + 2) = 2\varepsilon - 6.$$

(★1) By 1.11,  $0 \leq \varepsilon \leq 3$ .

Suppose that  $\varepsilon = 0$ .  $D + E$  has two (-2)-twigs (maximal twigs with each component a (-2) curve) with determinants  $a$  and  $b$ . By 1.10,  $a = b = 2$ . From (ii) we get  $n = -1$ , i.e.  $T_2^2 = 1$ . But then the proper transform of  $T_2$  in  $\overline{S}$  is a (-1)-curve, so  $D + E$  is not NC-minimal w.r.t.  $E$ .

The following four results follow formally from (\*) (without reference to  $q$ ).

(★2)  $n + 2 < 0$  if  $\varepsilon \leq 2$ .

(★3) Suppose that  $\varepsilon = 1$ . We have  $n + 2 = -1$  or  $-2$  or  $-4$ , so  $n = -3$  or  $-4$  or  $-6$ . But (\*\*) gives  $n \leq -7$ .

(★4) Suppose that  $\varepsilon = 2$ . From (\*\*\*) we get  $n + 2 = -1$  or  $-2$ . So  $n = -3$  or  $-4$ . But (\*) gives  $n \leq -6$ .

(★5) Suppose that  $\varepsilon = 3$ . From (\*\*) we obtain that  $n \leq -5$ . From (\*\*\*),  $(a - 2)(n + 2) = 0$ . It follows that  $a = 2$ .

Let  $T_3$  be the member of the system  $|T_1|$  passing through  $p_2$ . Since  $E_0$  is tangent to  $T_2$  at  $p_2$ ,  $E_0$  is transversal to  $T_3$  at  $p_2$ . Hence  $E_0$  meets  $T_3$  transversally at a point  $p_3 \neq p_2$ . After the first blowing up  $E_0$  meets the proper transform of  $T_2$ . It follows that the proper transform of  $T_3$  in  $\mathbb{P}^2$  is a good asymptote of  $U$ .

Now assume that  $q \in E_0$ .

Assume that  $E_0$  meets  $T_2$  also in a point  $p_2 \neq q$ . Then  $E_0 \cap T_1 = \{q\}$ . Hence  $E_0$  is tangent to  $T_1$  at  $q$ . It follows that  $E_0$  is transversal to  $T_2$  at  $q$ . Hence the local intersection of  $E_0$  with  $T_2$  at  $p_2$  equals  $b - 1$ . Suppose that  $b = 2$ . Then  $na = -2$ , hence  $a = 2$  and  $n = -1$ , i.e.,  $T_2^2 = 1$ . After the first blowing up at  $q$ ,  $E_0$  leaves  $T_2$ .  $T_2$  at this stage becomes a 0-curve which meets  $E_0$  once. The proper transform of the system  $|T_2|$  in  $\mathbb{P}^2$  is a good asymptote of  $U$ , a contradiction. So  $b \geq 3$ . It follows that  $E_0$  is tangent to  $T_2$  at  $p_2$ . It follows that  $\psi$  involves one sprouting contraction w.r.t.  $D$ . On the other hand,  $\psi$  now involves  $a + b - 1$  contractions on  $E$ . Computing  $K_{\overline{N}} \cdot (K_{\overline{N}} + E_0)$  as above we get again have (\*), hence also (\*\*), (\*\*\*) and (★1) to (★5).

Assume that  $\varepsilon = 0$ . From (\*\*\*) we get that  $n + 2$  divides  $-6$ . From (\*\*),  $n \leq -8$ . It follows that  $n = -8$ ,  $a = 3$ . It follows further that  $b = 13$ . The proper transform of  $T_1$  in  $\overline{S}$  is a tip of  $D + E$  and it is a (-3) curve.  $D + E$  also has a twig consisting of 11 (-2)-curves. The twig is created by



blowing up over  $p_2$ . Hence  $\sum e_i = \frac{1}{3} + \frac{11}{12} > 1$ , a contradiction in view of 1.10. The cases  $\varepsilon = 1, 2, 3$  we eliminate as above.

Assume that  $E_0$  meets  $T_1$  in a point  $p_1 \neq q$ . Then  $E_0 \cap T_2 = \{q\}$ . Since  $b \geq 2$ ,  $E_0$  is tangent to  $T_2$  at  $q$ . Hence  $E_0$  is transversal to  $T_1$  at  $q$ . Suppose that  $a = 2$ . Then the proper transform of  $T_1$  in  $\bar{S}$  is a  $(-1)$ -curve and it meets  $E$  once. Thus  $D + E$  is not NC-minimal w.r.t.  $E$ , a contradiction. Hence  $a \geq 3$ , i.e.  $E_0$  is tangent to  $T_1$  at  $p_1$ . As in the previous case  $\psi$  involves one sprouting contraction and  $a + b - 1$  contractions on  $E$ . Again the  $(*)$ - and  $(\star)$ -results hold.

Suppose that  $\varepsilon = 0$ . As above we get  $n = -8$ ,  $a = 3$  and  $b = 13$ . Also  $E_0^2 = 2a = 6$ . Let  $C_1, C_2$  be the two  $(-1)$ -components of  $D$  which meet  $E$ .  $D - (C_1 + C_2)$  has three connected components, two single curves  $D_1, D_2$  (they are tips of  $D + E$ ) and one chain  $R$ .  $D_1$  is the proper transform of the curve produced by the first blowing up over  $p_1$  and  $D_2$  is the proper transform of  $T_2$ .  $R$  is a chain which has the proper transform of  $T_1$  as a tip. It is a  $(-3)$ -curve. The rest of  $R$  consists of 12  $(-2)$ -curves. We have  $D_1^2 = -2$ ,  $D_2^2 = -5$ ,  $E^2 = -9$ .

**2.4.1** Let  $Q = D_1 + D_2 + R + E$ . Consider the surface

$$Y = \bar{S} \setminus Q.$$

We claim that  $\bar{\kappa}(Y) \geq 0$ . We have  $K_{\bar{S}} \cdot (K_{\bar{S}} + Q) = 2 + K_{\bar{S}} \cdot (K_{\bar{S}} + D + E) = 4 - \varepsilon = 4$ . Since  $Q$  has 4 components that are rational trees we find  $(2K_{\bar{S}} + Q) \cdot (K_{\bar{S}} + Q) = -\varepsilon = 0$ . By the Riemann-Roch Theorem,  $h^0(2K_{\bar{S}} + Q) + h^0(-K_{\bar{S}} - Q) > 0$ . Suppose that  $\bar{\kappa}(Y) = -\infty$ . Suppose  $|2K_{\bar{S}} + Q| = \emptyset$ . Then  $-K_{\bar{S}} - Q > 0$ .

**2.4.2** In view of 1.6, we have  $h^0(-K_{\bar{S}} - D - E) = 0$  or  $-K_{\bar{S}} - D - E = 0$ . Hence by the Riemann-Roch theorem and 1.7

$$h^0(2K_{\bar{S}} + D + E) \geq 1 + K_{\bar{S}} \cdot (K_{\bar{S}} + D + E) = 3 - \varepsilon \text{ or } K_{\bar{S}} = -D - E \text{ and } h^0(2K_{\bar{S}} + D + E) \geq 2 - \varepsilon.$$

Hence  $2K_{\bar{S}} + D + E \geq 0$ . We obtain that  $K_{\bar{S}} + C_1 + C_2 = 2K_{\bar{S}} + D + E + (-K_{\bar{S}} - Q) \geq 0$ . This gives  $K_{\bar{S}} \geq 0$ , a contradiction. Hence  $2K_{\bar{S}} + Q > 0$  and  $\bar{\kappa}(Y) \geq 0$ .

**2.4.3** We claim that the pair  $(\bar{S}, Q)$  is almost minimal. If it is not then there exists a  $(-1)$ -curve  $L$  such that  $L \subset \text{Supp}(K_{\bar{S}} + Q)^-$  and  $L$  is not a component of  $Q$ . But the intersection matrix of  $Q$  is negative definite and all irreducible components of  $Q$  are components of  $(K_{\bar{S}} + Q)^-$ . Since the rank of  $\text{Pic}(\bar{S})$  equals the number of irreducible components of  $Q$  plus 1 we reach contradiction with 1.11.

Since  $\chi(Y) = -1$ , the BMY-inequality (Langer's version, see 1.13) gives

$$\frac{1}{d(D_1)} + \frac{1}{d(D_2)} + \frac{1}{d(R)} + \frac{1}{d(E)} \geq 1.$$

This is a contradiction since  $d(D_1) = 2, d(D_2) = 5, d(R) = 27, d(E) = 9$ .

The cases  $\varepsilon = 1, 2$  we eliminate as above. If  $\varepsilon = 3$  we get, as above, that  $a = 2$ , but we already proved that  $a \geq 3$ .

Assume that  $E_0 \cap T = \{q\}$ . Then  $E_0$  is singular, which we have seen is not the case.  $\square$

**Corollary 2.5.** *Let  $t$  denote the number of sprouting contractions in  $\psi$ , see 2.2. Then*

$$7 + t \geq 2\varepsilon + \gamma.$$

*Proof.* This follows from 2.2(b) and 2.4.  $\square$

### 3. SEPARATION OF BRANCHES I: THE BRANCHES ARE TANGENT AT INFINITY

**3.1.** Let  $\lambda, \tilde{\lambda}$  be the branches of  $\bar{U}$  at  $L_\infty$ . The resolution process  $\Phi$ , see 0.1, can be described in terms of Hamburger-Noether (HN-) pairs. For the definition in our context and basic properties of HN-pairs we refer to [C-NKR, 1.12]; see also [KR1, Appendix] or [Ru1]. We remark also that to each HN-pair there is tacitly associated an  $a \in \mathbb{C}$ , a parameter that determines the location of the branch on the last exceptional curve produced by the blowups prescribed by the pair. Let  $(\frac{c_1}{p_1}), \dots, (\frac{c_h}{p_h})$  (resp.  $(\frac{\tilde{c}_1}{\tilde{p}_1}), \dots, (\frac{\tilde{c}_h}{\tilde{p}_h})$ ) be the sequence of HN-pairs of  $\lambda$  (resp.  $\tilde{\lambda}$ ). We recall that, by definition,  $c_1 = \lambda \cdot L_\infty$ ,  $\tilde{c}_1 = \tilde{\lambda} \cdot L_\infty$ ,  $c_{i+1} = \text{GCD}(c_i, p_i)$  and  $c_i \geq p_i$ . Let  $\mu_1, \mu_2, \dots$  (resp.

$\tilde{\mu}_1, \tilde{\mu}_2, \dots$ ) be the sequence of multiplicities of all singular points of  $\lambda$  infinitely near  $\lambda \cap L_\infty$  (resp. of  $\tilde{\lambda}$  infinitely near  $\tilde{\lambda} \cap L_\infty$ ).

**3.1.1** Then

$$(i) \quad \sum_{i \geq 1} \mu_i = c_1 + p_1 + p_2 + \dots + p_h - 1.$$

$$(ii) \quad \sum_{i \geq 1} \mu_i^2 = c_1 p_1 + c_2 p_2 + \dots + c_h p_h.$$

$$(iii) \quad \sum_{i \geq 1} \tilde{\mu}_i = \tilde{c}_1 + \tilde{p}_1 + \tilde{p}_2 + \dots + \tilde{p}_h - 1.$$

$$(iv) \quad \sum_{i \geq 1} \tilde{\mu}_i^2 = \tilde{c}_1 \tilde{p}_1 + \tilde{c}_2 \tilde{p}_2 + \dots + \tilde{c}_h \tilde{p}_h.$$

Throughout this section we assume that  $\lambda \cap L_\infty = \tilde{\lambda} \cap L_\infty = q$  and that the branches cannot be separated by an automorphism of  $\mathbb{C}^2$ . At the end we will come to a contradiction. We will also assume that the resolution tree  $D'$  has the smallest possible number of irreducible components, i.e., if  $\sigma: \mathbb{C}^2 \rightarrow \mathbb{C}^2$  is an automorphism, then the number components of the resolution tree of  $\sigma(\overline{U})$  is not less than the number of components of  $D'$ .

**3.1.2** Let  $s$  denote the number of *common pairs* of  $\lambda$  and  $\tilde{\lambda}$ . By this we mean that

$$\frac{c_i}{p_i} = \frac{\tilde{c}_i}{\tilde{p}_i} \quad \text{and} \quad a_i = \tilde{a}_i \quad \text{for} \quad i = 1, \dots, s,$$

but one of these conditions is violated for  $i = s + 1$ . Then the branches separate somewhere along the chains created by the pairs  $\binom{c_{s+1}}{p_{s+1}}, \binom{\tilde{c}_{s+1}}{\tilde{p}_{s+1}}$ . Let  $m_1, m_2, \dots$  be the sequence of multiplicities of all singular points of  $\overline{U}$  infinitely near to the point  $q$ .

**3.1.3** We have the following formulas, see [KR], Appendix.

$$(i) \quad \sum m_i = c_1 + \sum p_i - 1 + \tilde{c}_1 + \sum \tilde{p}_i - 1.$$

$$(ii) \quad \sum m_i^2 = \sum_{i=1}^s (p_i + \tilde{p}_i)(c_i + \tilde{c}_i) + \sum_{i>s} p_i c_i + \sum_{i>s} \tilde{p}_i \tilde{c}_i + 2 \min(\tilde{p}_{s+1} c_{s+1}, p_{s+1} \tilde{c}_{s+1}).$$

**Lemma 3.2.** Let  $\gamma' = E'^2$ , see 0.1. We obtain the following formulas.

$$(a) \quad \gamma' + 2d = \sum p_i + \sum \tilde{p}_i.$$

$$(b) \quad \gamma' + d^2 = \sum_{i=1}^s (p_i + \tilde{p}_i)(c_i + \tilde{c}_i) + \sum_{i>s} p_i c_i + \sum_{i>s} \tilde{p}_i \tilde{c}_i + 2 \min(\tilde{p}_{s+1} c_{s+1}, p_{s+1} \tilde{c}_{s+1}).$$

where  $d = c_1 + \tilde{c}_1$ .

*Proof.* (a) We have  $K_{\mathbb{P}^2} \cdot \overline{U} = -3d$ ,  $K_{\overline{S}'} \cdot E' = -2 + \gamma'$ . Blowing up a point of multiplicity  $m$  of a curve  $X$  increases the quantity  $K \cdot X$  by  $m$ . Hence  $K_{\overline{S}'} \cdot E' - K_{\mathbb{P}^2} \cdot \overline{U} = -2 + \gamma' + 3d = \sum m_i$ . The statement follows from (i) above.

(b) We have  $\overline{U}^2 - E'^2 = d^2 + \gamma' = \sum m_i^2$ . The statement follows now from (ii).  $\square$



**3.2.1** We put

$$c_1 - p_1 = \alpha c_2, \quad \tilde{c}_1 - \tilde{p}_1 = \tilde{\alpha} \tilde{c}_2 \quad \text{and} \quad \alpha_0 = \min(\alpha, \tilde{\alpha}).$$

Since  $2 \min(\tilde{p}_{s+1} c_{s+1}, p_{s+1} \tilde{c}_{s+1}) \leq \tilde{p}_{s+1} c_{s+1} + p_{s+1} \tilde{c}_{s+1}$ , 3.2(b) gives

$$\gamma' + d^2 \leq \sum_{i=1}^s (p_i + \tilde{p}_i)(c_i + \tilde{c}_i) + \sum_{i>s} p_i c_i + \sum_{i>s} \tilde{p}_i \tilde{c}_i + \tilde{p}_{s+1} c_{s+1} + p_{s+1} \tilde{c}_{s+1}$$

and

$$\gamma' + d^2 \leq (p_1 + \tilde{p}_1)d + (c_2 + \tilde{c}_2)(P + \tilde{P}),$$

where  $P = \sum_{i \geq 2} p_i$ ,  $\tilde{P} = \sum_{i \geq 2} \tilde{p}_i$ .

From this

$$(i) \quad d(d - p_1 - \tilde{p}_1) + \gamma' \leq (c_2 + \tilde{c}_2)(P + \tilde{P}).$$

Since  $d - p_1 - \tilde{p}_1 = c_1 - p_1 + \tilde{c}_1 - \tilde{p}_1 \geq \alpha_0(c_2 + \tilde{c}_2)$  and since  $\gamma' \geq 1$  we get

$$(ii) \quad \alpha_0 d < P + \tilde{P}.$$

**Lemma 3.3.** *Let  $h_\Phi$  (resp.  $h_\Psi$ ) be the number of sprouting contractions in  $\Phi$  (resp.  $\Psi$ ). Then  $h_\Phi = 6 - K_{\overline{S}} \cdot (K_{\overline{S}} + D) = 2 + \varepsilon + \gamma + h_\Psi$ . If  $E'$  is not touched by the contractions in  $\Psi$ , then  $\gamma = \gamma'$ . If, moreover,  $L'_\infty$ , the proper transform of  $L_\infty$  in  $\overline{S}'$ , is not a  $(-1)$ -curve, then  $h_\Psi = 0$ .*

*Proof.* Under a subdivisional blowing up of a point on a divisor  $T$  the quantity  $K \cdot (K + T)$  doesn't change. Under a sprouting blowing up the quantity decreases by 1. Hence  $h_\Phi = K_{\mathbb{P}^2} \cdot (K_{\mathbb{P}^2} + L_\infty) - K_{\overline{S}'} \cdot (K_{\overline{S}'} + D')$  and  $h_\Psi = K_{\overline{S}} \cdot (K_{\overline{S}} + D) - K_{\overline{S}'} \cdot (K_{\overline{S}'} + D')$ . Since  $K_{\overline{S}} \cdot E = -2 + \gamma$  the result follows from 1.7.  $\square$

**Lemma 3.4.** *We have  $s = 0$ .*

*Proof.* Suppose that  $s \geq 1$ . Note that then  $\lambda$  and  $\tilde{\lambda}$  are both tangent to  $L_\infty$ . (Otherwise both are not tangent to  $L_\infty$  and  $q$  is a point of multiplicity  $\deg(\overline{U})$  on  $\overline{U}$ .) Hence  $c_1 > p_1, \tilde{c}_1 > \tilde{p}_1$ . Also, both branches have more than one characteristic pair, i.e.,  $h > 1$  and  $\tilde{h} > 1$ . We put

$$\frac{c_1}{c_2} = k = \frac{\tilde{c}_1}{\tilde{c}_2}, \quad \text{and} \quad \frac{p_1}{c_2} = l = \frac{\tilde{p}_1}{\tilde{c}_2}.$$

We have  $\alpha = \tilde{\alpha} = k - l \geq 1$ .

Suppose that  $\alpha = 1$ , i.e.,  $k = l + 1$ . The blowing up over  $q$  according to the pair  $\binom{l+1}{l}$  produces a chain  $L + C + M$ , where  $L$  has  $l$  components with  $L_\infty$  as a  $(-1)$ -tip,  $C$  is the last exceptional curve and  $M$ , a  $(-l - 1)$ -curve, is (the proper transform of) the first exceptional curve. The branches  $\lambda, \tilde{\lambda}$  have common center  $q'$  on  $C \setminus (L \cup M)$ . In  $\Phi^{-1}$  we now blow up  $q'$ . Let  $A$  be the resulting exceptional curve. Let us perform  $l - 1$  successive additional sprouting blowups (they will not be part of  $\Phi^{-1}$ ), starting with a point on  $A$  that is *not* the center of  $\lambda$  or  $\tilde{\lambda}$ , creating a chain  $A + B$  attached to  $C$ , with  $B$  of length  $l - 1$ . Let  $L_\infty^\dagger$  be the last exceptional curve. As it is well known, we can now blow down, beginning with  $L_\infty$ , the curves in  $L$ , then  $C$ , then  $A + (B - L_\infty^\dagger)$ , then  $M$ , producing a new completion of  $\mathbb{C}^2$  with  $L_\infty^\dagger$  as new line at infinity and a new completion  $\overline{U}^\dagger$  of  $U$ .

**3.4.1** Let us note for further reference that we have performed an *elementary transformation* of  $\mathbb{C}^2$  determined by  $q \in L_\infty$ , the pair  $\binom{l+1}{l}$ , the choice of  $q' \in C$  and the choice of further fundamental points in creating the chain  $B$ .

The task of producing the NC-resolution for  $\overline{U} + L_\infty$  is accomplished by further blowups over  $A$ . Let  $A^\sharp$  be the resulting configuration of curves and put

$$\Gamma = L + M + C + A^\sharp + B.$$

Then, as a set,  $D' = L + C + M + A^\sharp \subset \Gamma$ . In constructing the minimal NC-resolution of  $\overline{U}^\dagger + L_\infty^\dagger$  we have to reconstruct  $A$ , hence also  $B$ , and then  $A^\sharp$ . Hence also  $D^\dagger \subset \Gamma$ . There are three

possibilities.

(i) The centers of  $\lambda$  and  $\tilde{\lambda}$  on  $A$  are not on  $C$  and  $l = 1$ . Then  $D^\dagger = A^\sharp$ . (We have  $L_\infty^\dagger = A$ .)

(ii) The centers of  $\lambda$  and  $\tilde{\lambda}$  on  $A$  are not on  $C$  and  $l > 1$ . Then  $D^\dagger = B + A^\sharp + M$ .

(iii) The center of  $\lambda$  or  $\tilde{\lambda}$  on  $A$  is on  $C$ . Then  $D^\dagger = B + A^\sharp + M + C$ .

In each case  $D^\dagger$  has fewer components than  $D'$ , contrary to our assumption.

Hence  $\alpha \geq 2$ . Then either  $L_\infty'^2 \leq -2$  or  $D'$  has a twig with an initial chain  $L_\infty' + L$  with  $L_\infty'^2 = -1$ ,  $L$  a  $(-2)$ -chain attached to a  $(\leq -3)$ -curve in  $D'$  and  $E \cdot (L_\infty' + L) = 0$ . This implies that  $E'$  is not touched by the contractions in  $\Psi: \overline{S}' \rightarrow \overline{S}$ . Hence  $\gamma' = \gamma$ .

By 2.5,  $\gamma \leq 9$ . Suppose that  $\alpha \geq 3$ . We have  $3d < P + \tilde{P}$  by 3.2.1(ii). From 3.2(a) we obtain  $P + \tilde{P} + p_1 + \tilde{p}_1 - \gamma = 2d$ . We get that  $d + p_1 + \tilde{p}_1 < \gamma \leq 9$ . Thus  $d < 9 - p_1 - \tilde{p}_1 \leq 7$ . But  $d = c_1 + \tilde{c}_1 \geq 2(\alpha + 1) \geq 8$ , a contradiction.

Hence  $\alpha = 2$ , i.e.,  $k = l + 2$ . Since  $GCD(k, l) = 1$ ,  $l$  and  $k$  are odd. We have  $d - p_1 - \tilde{p}_1 = 2(c_2 + \tilde{c}_2)$ . Substitute this into 3.2.1(i). We obtain

$$2d(c_2 + \tilde{c}_2) + \gamma \leq (c_2 + \tilde{c}_2)(P + \tilde{P})$$

and

$$(*) \quad (c_2 + \tilde{c}_2)(2d - P - \tilde{P}) + \gamma \leq 0.$$

By 3.2(a),  $2d - P - \tilde{P} = p_1 + \tilde{p}_1 - \gamma$ . Hence

$$(c_2 + \tilde{c}_2)(l(c_2 + \tilde{c}_2) - \gamma) + \gamma \leq 0.$$

From this  $l(c_2 + \tilde{c}_2) < \gamma$ . Thus  $l(c_2 + \tilde{c}_2) \leq 8$ , which implies  $l \leq 4$ . Hence  $l \leq 3$ .

Suppose that  $l = 3$ . Then  $c_2 + \tilde{c}_2 = 2$  and we have  $2(6 - \gamma) + \gamma \leq 0$  which gives  $\gamma \geq 12$ , a contradiction.

Therefore  $l = 1$  and  $k = 3$ .  $(*)$  takes the form  $(c_2 + \tilde{c}_2)(c_2 + \tilde{c}_2 - \gamma) + \gamma \leq 0$ . By simple algebra we get  $(c_2 + \tilde{c}_2 - 1)(c_2 + \tilde{c}_2 + 1 - \gamma) \leq -1$ . This implies that

$$(**) \quad c_2 + \tilde{c}_2 = p_1 + \tilde{p}_1 \leq \gamma - 2.$$

The proper transform of  $L_\infty$  in  $\overline{S}$  is a  $(-2)$ -curve. Hence  $D' = D$ . Blowing up on  $L_\infty$  according to the pair  $\binom{c_1}{p_1}$  produces a chain  $L_\infty' + C + M$ , where  $M$  consists of two  $(-2)$ -curves and  $C$  is branching in  $D$ .  $L_\infty'$  and  $M$  are maximal twigs in  $D + E$  and contribute  $\frac{1}{2} + \frac{2}{3} > 1$  to  $\sum e_i$ . In view of 1.10 and 2.5,

$$(***) \quad \varepsilon \geq 1 \quad \text{and} \quad \gamma \leq 7 + t - 2\varepsilon \leq 5 + t.$$

Hence  $\gamma \leq 7$ . If  $\gamma = 7$ , then  $\varepsilon = 1$  and  $t = 2$ , so  $D + E$  has at least two maximal twigs with  $(-2)$ -tips and not contained in  $L$ , those produced by the pairs  $\binom{c_h}{p_h}$  and  $\binom{\tilde{c}_h}{\tilde{p}_h}$ . In view of 1.3.1 we get that  $\sum e_i > 2$ , in contradiction to 1.10.

Hence we have  $\gamma \leq 6$ , so  $c_2 + \tilde{c}_2 \leq 4$ .

Suppose that  $c_2 = \tilde{c}_2 = 2$ . Let  $a$  be the number of common pairs of type  $\binom{2}{2}$ . Thus  $s \geq a + 1$ . If  $s \geq a + 2$  then the next common pair is of type  $\binom{2}{1}$ , followed by a number, possibly zero, of common pairs of type  $\binom{1}{1}$ . The branches then both meet  $D$  transversally at different points of the last  $(-1)$ -curve. It follows that  $t = 0$  and  $(***)$  gives  $\gamma \leq 5$ . We reach a contradiction with  $(**)$ .

Hence  $s = a + 1$ . Then either  $\binom{c_{s+1}}{p_{s+1}} = \binom{\tilde{c}_{s+1}}{\tilde{p}_{s+1}}$ , so either  $\binom{2}{2}$  or  $\binom{2}{1}$ , or, say,  $\binom{c_{s+1}}{p_{s+1}} = \binom{2}{1}$ ,  $\binom{\tilde{c}_{s+1}}{\tilde{p}_{s+1}} = \binom{2}{2}$ . Hence  $m = \min(c_{s+1}\tilde{p}_{s+1}, \tilde{c}_{s+1}p_{s+1}) = 2$  or  $4$ . We have also  $d = c_1 + \tilde{c}_1 = 3(c_2 + \tilde{c}_2) = 12$ . Note that  $\binom{c_h}{p_h} = \binom{\tilde{c}_h}{\tilde{p}_h} = \binom{2}{1}$ . Hence in 3.2(b) we have  $c_h p_h + \tilde{c}_h \tilde{p}_h = 2 + 2$  and all other individual terms on the RHS and the term  $d^2$  on the LHS are divisible by 4. Hence  $\gamma$  is divisible by 4, so  $\gamma = 4$ , and we have a contradiction with  $(**)$ .

Suppose now that  $c_2 = 1$ . Then  $\tilde{c}_2 \leq 3$ . Let  $a$  be the number of common pairs of type  $\binom{\tilde{c}_2}{c_2}$ . Then  $s = 1 + a$ . We have  $h = s + 1$  with  $p_h = c_h = 1$  and  $\tilde{h} = 1 + a + b + 1$ , where  $a + b$  is the

total number of pairs equal to  $\binom{\tilde{c}_2}{\tilde{c}_2}$ . We have  $m = \min(c_{s+1}\tilde{p}_{s+1}, \tilde{c}_{s+1}p_{s+1}) = \tilde{p}_{s+1}$ . The formulas 3.2 take the form

$$(1) \quad \gamma + 2(3 + 3\tilde{c}_2) = s + 1 + (s + b)\tilde{c}_2 + \tilde{p}_h.$$

and

$$(2) \quad \gamma + (3 + 3\tilde{c}_2)^2 = (3 + 3\tilde{c}_2)(1 + \tilde{c}_2) + (s - 1)(1 + \tilde{c}_2)^2 + 2\tilde{p}_{s+1} + 1 + b\tilde{c}_2^2 + \tilde{c}_2\tilde{p}_h.$$

Suppose that  $\tilde{c}_2 = 3$ . Then  $\gamma = 6$  by (\*\*). By (\*\*),  $t \geq 1$ . This implies that  $\tilde{c}_{s+1} = \tilde{p}_{s+1} = 3$  and  $\tilde{p}_h = 1$ . (1) and (2) now give  $6 + 24 = s + 1 + 3(s + b) + 1$ , i.e.,  $28 = 4s + 3b$  and  $6 + 144 = 48 + (s - 1)16 + 9b + 10$ , i.e.,  $108 = 16s + 9b$ . The system of equations has no integer solutions.

Before proceeding with the analysis of cases we note the following. Since  $\kappa(K_{\overline{S}} + E) = -\infty$  by [Kor], we have  $h^0(2K_{\overline{S}} + E) = 0$  and by the Riemann-Roch theorem

$$(\#) \quad h^0(-K_{\overline{S}} - E) \geq K_{\overline{S}} \cdot (K_{\overline{S}} + E) = K_{\overline{S}}^2 - 2 + \gamma.$$

As argued in 2.4.2

$$(\#\#) \quad h^0(2K_{\overline{S}} + D + E) \geq 1 + K_{\overline{S}} \cdot (K_{\overline{S}} + D + E) = 3 - \epsilon \quad \text{or} \quad K_{\overline{S}} = -D - E.$$

Now suppose that  $\tilde{c}_2 = 2$ . (1) and (2) give  $\gamma + 18 = s + 1 + (s + b)2 + 1$ , i.e.,  $\gamma + 16 = 3s + 2b$  and  $\gamma + 81 = 27 + (s - 1)9 + 4b + 3 + 2\tilde{p}_{s+1}$ , i.e.,  $\gamma + 60 = 9s + 4b + 2\tilde{p}_{s+1}$ . We have  $5 \leq \gamma \leq 6$  and  $\tilde{p}_{s+1} \leq 2$ . We find two solutions:

- (i)  $\gamma = 5, s = 7, b = 0, \tilde{p}_{s+1} = 1$
- (ii)  $\gamma = 6, s = 6, b = 2, \tilde{p}_{s+1} = 2$ .

In case (i) we find  $K_{\overline{S}}^2 = 10 - b_2(\overline{S}) = -2$  and  $\varepsilon = 1$ . By (#) and (\#\#),  $-K_{\overline{S}} - E \geq 0$  and  $2K_{\overline{S}} + D + E \geq 0$ . We obtain  $K_{\overline{S}} + D = 2K_{\overline{S}} + D + E + (-K_{\overline{S}} - E) \geq 0$ , a contradiction. In case (ii) we have  $b_2(\overline{S}) = 13$ , so  $K_{\overline{S}}^2 = -3$ , and  $\varepsilon = 1$ . We come to a contradiction by the same argument.

Suppose that  $\tilde{c}_2 = 1$ . Then  $d = 6$ . The formulas give  $\gamma + 10 = 2s + b$  and  $\gamma + 24 = 4s + b$ . We get the solution

$$(iii) \quad \gamma = 4, s = 7.$$

We find  $b_2(\overline{S}) = 11$ ,  $K_{\overline{S}}^2 = -1$ ,  $\varepsilon = 2$ . Clearly  $K_{\overline{S}} \neq -D - E$ , e.g.,  $L'_\infty \cdot K_{\overline{S}} = 0$ ,  $L'_\infty \cdot (-D - E) = 1$ . We come to a contradiction by the same argument. □

**3.5.** We have shown that  $s = 0$ . Suppose that the branches stay together after the first blowing up, i.e., they both are tangent to  $L_\infty$ . Let, as in 3.2.1,

$$c_1 - p_1 = \alpha c_2, \quad \tilde{c}_1 - \tilde{p}_1 = \tilde{\alpha} \tilde{c}_2.$$

We will show that, possibly at the cost of increasing the number of components of  $D'$ , this case can be reduced to the case  $\alpha = \tilde{\alpha} = 1$  and  $\frac{c_1}{p_1} = \frac{\tilde{c}_1}{\tilde{p}_1} = \frac{l+1}{l}$ . This case will be dealt with in 3.6.

Suppose that  $\tilde{\alpha} = 1$ . Let  $\tilde{p}_1 = l\tilde{c}_2$ . Then  $\tilde{c}_1 = (l + 1)\tilde{c}_2$ . We use the notation of 3.4.1. After blowing up according to  $\binom{l+1}{l}$ , the center  $\tilde{p}$  of  $\tilde{\lambda}$  is on  $C \setminus (L \cup M)$ . We now have three possibilities.

- (i) The center  $p$  of  $\lambda$  is on  $M$ . Equivalently,  $p_1 > l(c_1 - p_1)$ .
- (ii) The center  $p$  of  $\lambda$  is on  $C \setminus (L \cup M)$ . Equivalently,  $p_1 = l(c_1 - p_1)$ , or  $\alpha = 1$ . Moreover,  $p \neq p'$ .
- (iii) The center  $p$  of  $\lambda$  is on  $L$ . Equivalently,  $p_1 < l(c_1 - p_1)$ .

Suppose we have (i) or (ii). We then perform an elementary transformation exactly as in 3.4.1 with  $q' = \tilde{p}$ . The argument in the proof of 3.4, slightly modified, shows that we obtain a completion with smaller  $D'$ .

**3.5.1** We remark that if the HN-sequence for  $\tilde{\lambda}$  has at least  $l$  pairs  $\binom{\tilde{c}_2}{\tilde{c}_2}$  following  $\binom{\tilde{c}_1}{\tilde{p}_1}$ , or if  $\tilde{c}_2 = 1$ , we can construct the above elementary transformation with blowups following  $\tilde{\lambda}$ , and it

will then separates the branches. Hence this is not the case.

Suppose we have (iii). We now perform an elementary transformation as above, but with  $q' \neq \tilde{p}$ . Then we are in situation (ii) w.r.t. the new coordinate system, i.e., we have  $\alpha = \tilde{\alpha} = 1$ .

We may assume that  $\alpha \geq 2$ ,  $\tilde{\alpha} \geq 2$ . We write 3.2(b) as

$$\gamma + \alpha c_1 c_2 + \tilde{\alpha} \tilde{c}_1 \tilde{c}_2 + 2c_1 \tilde{c}_1 = \sum_{i \geq 2} p_i c_i + \sum_{i \geq 2} \tilde{p}_i \tilde{c}_i + 2 \min(c_1 \tilde{p}_1, \tilde{c}_1 p_1).$$

We have  $2c_1 \tilde{c}_1 = \tilde{c}_1(p_1 + \alpha c_2) + c_1(\tilde{p}_1 + \tilde{\alpha} \tilde{c}_2) = \tilde{c}_1 p_1 + c_1 \tilde{p}_1 + \alpha c_2 \tilde{c}_1 + \tilde{\alpha} c_1 \tilde{c}_2$ .  
Therefore

$$\gamma + \tilde{c}_1 p_1 + c_1 \tilde{p}_1 + \alpha c_2 \tilde{c}_1 + \tilde{\alpha} c_1 \tilde{c}_2 + \alpha c_1 c_2 + \tilde{\alpha} \tilde{c}_1 \tilde{c}_2 = \sum_{i \geq 2} p_i c_i + \sum_{i \geq 2} \tilde{p}_i \tilde{c}_i + 2 \min(c_1 \tilde{p}_1, \tilde{c}_1 p_1)$$

and

$$\gamma + \tilde{c}_1 p_1 + c_1 \tilde{p}_1 - 2 \min(c_1 \tilde{p}_1, \tilde{c}_1 p_1) + (c_1 + \tilde{c}_1)(\alpha c_2 + \tilde{\alpha} \tilde{c}_2) = \sum_{i \geq 2} p_i c_i + \sum_{i \geq 2} \tilde{p}_i \tilde{c}_i.$$

Let  $\beta = \tilde{c}_1 p_1 + c_1 \tilde{p}_1 - 2 \min(c_1 \tilde{p}_1, \tilde{c}_1 p_1) \geq 0$ . We get

$$(*) \quad \gamma + \beta + (c_1 + \tilde{c}_1)(\alpha c_2 + \tilde{\alpha} \tilde{c}_2) = \sum_{i \geq 2} p_i c_i + \sum_{i \geq 2} \tilde{p}_i \tilde{c}_i.$$

From 3.2(a) we get

$$(**) \quad \gamma + c_1 + \tilde{c}_1 + \alpha c_2 + \tilde{\alpha} \tilde{c}_2 = \sum_{i \geq 2} p_i + \sum_{i \geq 2} \tilde{p}_i.$$

We may assume by symmetry that  $c_2 \geq \tilde{c}_2$ . Multiply (\*\*) by  $c_2$  and subtract (\*). We obtain

$$\gamma c_2 + (c_1 + \tilde{c}_1)c_2 + c_2(\alpha c_2 + \tilde{\alpha} \tilde{c}_2) \geq \gamma + \beta + (c_1 + \tilde{c}_1)(\alpha c_2 + \tilde{\alpha} \tilde{c}_2).$$

So

$$\gamma(c_2 - 1) + (c_1 + \tilde{c}_1)c_2 \geq \beta + (\alpha c_2 + \tilde{\alpha} \tilde{c}_2)(c_1 + \tilde{c}_1 - c_2).$$

Since  $\alpha \geq 2, \tilde{\alpha} \geq 2, \beta \geq 0$  we have

$$\gamma(c_2 - 1) + (c_1 + \tilde{c}_1)c_2 \geq (2c_2 + 2)(c_1 + \tilde{c}_1 - c_2).$$

From this

$$\gamma(c_2 - 1) \geq (2c_2 + 2)(c_1 + \tilde{c}_1) - (2c_2 + 2)c_2 - (c_1 + \tilde{c}_1)c_2.$$

So

$$\gamma(c_2 - 1) \geq (c_2 + 2)(c_1 + \tilde{c}_1) - 2(c_2 + 1)c_2.$$

We have  $\gamma \leq 9$  by 2.5,  $\tilde{c}_1 \geq 3$  since  $\tilde{\alpha} \geq 2$ . Also  $c_1 \geq 3c_2$  since  $\alpha \geq 2$ . We obtain

$$9c_2 - 9 \geq (c_2 + 2)(3c_2 + 3) - 2(c_2 + 1)c_2.$$

We get

$$0 \geq c_2^2 - 2c_2 + 15.$$

This is a contradiction.

**3.6.** In this section we temporarily drop the assumption that  $D'$  has the smallest possible number of components. We consider here the case  $s = 0$ ,  $c_1 = (l + 1)c_2$ ,  $p_1 = lc_2$ ,  $\tilde{c}_1 = (l + 1)\tilde{c}_2$ ,  $\tilde{p}_1 = l\tilde{c}_2$ . We will prove that this case does not occur. Suppose opposite. Let  $H'$  denotes the (-1)-curve produced by the pair  $\binom{c_1}{p_1}$  and let  $H = \Psi(H')$ . The branches meet  $H'$  in two different points.  $\Psi$  involves  $l$  successive contractions beginning with  $L'_\infty$ .  $H'$  is not contracted by  $\Psi$ . Let  $F$  (resp.  $\tilde{F}$ ) denotes the part of  $D'$  produced by the pairs  $\binom{c_2}{p_2}, \dots, \binom{c_h}{p_h}$  (resp.  $\binom{\tilde{c}_2}{\tilde{p}_2}, \dots, \binom{\tilde{c}_h}{\tilde{p}_h}$ ). Let  $C$  (resp.  $\tilde{C}$ ) be the unique (-1)-curve in  $F$  (resp.  $\tilde{F}$ )

Let  $r$  (resp.  $\tilde{r}$ ) denotes the number of pairs equal to  $\binom{c_2}{c_2}$  (resp.  $\binom{\tilde{c}_2}{\tilde{c}_2}$ ). Hence  $p_{r+2} < c_{r+2} = c_2$  and  $c_i \leq \frac{1}{2}c_2$  for  $i > r+2$ . We put  $P' = \sum_{i \geq r+2} p_i$ . In similar way we define  $\tilde{P}'$ . Notice that

$c_2 > 1, \tilde{c}_2 > 1$  by the argument in 3.5.1. Therefore  $h > r+1, \tilde{h} > \tilde{r}+1$ , i.e.,  $P' \geq 1$  and  $\tilde{P}' \geq 1$ . Again by 3.5.1 we have  $r \leq l-1, \tilde{r} \leq l-1$ .

**3.6.1** We note that  $D+E$  has at least 3 maximal twigs, the  $-(l+1)$ -curve  $M$  (see 3.4.1), and one each in  $F$  an  $\tilde{F}$  with a  $\geq (-c_2)$ - and a  $\geq (-\tilde{c}_2)$ -curve as tip respectively. By 1.2.1 they contribute at least  $e = \frac{1}{u+1} + \frac{1}{c_2} + \frac{1}{\tilde{c}_2}$  to  $\sum e_i$  in 1.10. In particular,  $\varepsilon > 0$  if  $e > 1$ .

From 3.2(b) we get

$$\gamma + d^2 = (c_1 + \tilde{c}_1)(p_1 + \tilde{p}_1) + \sum_{i \geq 2} p_i c_i + \sum_{i \geq 2} \tilde{p}_i \tilde{c}_i = d(p_1 + \tilde{p}_1) + r c_2^2 + p_{r+2} c_2 + \tilde{r} \tilde{c}_2^2 + \tilde{p}_{\tilde{r}+2} \tilde{c}_2 + \sum_{i \geq r+3} p_i c_i + \sum_{i \geq \tilde{r}+3} \tilde{p}_i \tilde{c}_i.$$

From this

$$\gamma + d(c_2 + \tilde{c}_2) \leq r c_2^2 + \tilde{r} \tilde{c}_2^2 + c_2 p_{r+2} + \frac{1}{2} c_2 (P' - p_{r+2}) + \tilde{c}_2 \tilde{p}_{\tilde{r}+2} + \frac{1}{2} \tilde{c}_2 (\tilde{P}' - \tilde{p}_{\tilde{r}+2}).$$

From 3.2(a) we get

$$d = \frac{p_1 + \tilde{p}_1 + r c_2 + \tilde{r} \tilde{c}_2 + P' + \tilde{P}' - \gamma}{2}.$$

Hence

$$\gamma + \frac{1}{2}(c_2 + \tilde{c}_2)(p_1 + \tilde{p}_1 + r c_2 + \tilde{r} \tilde{c}_2 + P' + \tilde{P}') - \frac{1}{2} \gamma (c_2 + \tilde{c}_2) \leq r c_2^2 + \tilde{r} \tilde{c}_2^2 + \frac{1}{2} c_2 p_{r+2} + \frac{1}{2} c_2 P' + \frac{1}{2} \tilde{c}_2 \tilde{p}_{\tilde{r}+2} + \frac{1}{2} \tilde{c}_2 \tilde{P}'.$$

From this

$$(*) \quad (c_2 + \tilde{c}_2)(p_1 + \tilde{p}_1 + r c_2 + \tilde{r} \tilde{c}_2) + c_2 \tilde{P}' + \tilde{c}_2 P' - \gamma (c_2 + \tilde{c}_2) < 2r c_2^2 + 2\tilde{r} \tilde{c}_2^2 + c_2 p_{r+2} + \tilde{c}_2 \tilde{p}_{\tilde{r}+2}.$$

Since  $p_1 = l c_2, \tilde{p}_1 = l \tilde{c}_2$  and since  $P' \geq 1, \tilde{P}' \geq 1$  we have

$$(c_2 + \tilde{c}_2)(l(c_2 + \tilde{c}_2) + r c_2 + \tilde{r} \tilde{c}_2) < 2r c_2^2 + 2\tilde{r} \tilde{c}_2^2 + c_2 p_{r+2} + \tilde{c}_2 \tilde{p}_{\tilde{r}+2} + (\gamma - 1)(c_2 + \tilde{c}_2),$$

$$l(c_2 + \tilde{c}_2)^2 + (r + \tilde{r})c_2 \tilde{c}_2 < r c_2^2 + \tilde{r} \tilde{c}_2^2 + c_2 p_{r+2} + \tilde{c}_2 \tilde{p}_{\tilde{r}+2} + (\gamma - 1)(c_2 + \tilde{c}_2).$$

Since  $r, \tilde{r} \leq l-1$  and  $p_{r+2} \leq c_2 - 1, \tilde{p}_{\tilde{r}+2} \leq \tilde{c}_2 - 1$

$$l(c_2 + \tilde{c}_2)^2 + (r + \tilde{r})c_2 \tilde{c}_2 < l(c_2^2 + \tilde{c}_2^2) + (\gamma - 2)(c_2 + \tilde{c}_2).$$

Finally

$$(**) \quad c_2 \tilde{c}_2 (2l + r + \tilde{r}) < (\gamma - 2)(c_2 + \tilde{c}_2).$$

Suppose that  $l \geq 3$ . Then  $6c_2 \tilde{c}_2 < 7(c_2 + \tilde{c}_2)$ . This implies  $c_2 = \tilde{c}_2 = 2$ . But then  $\varepsilon > 0$  by 3.6.1. This implies  $\gamma \leq 7$  by 2.5. Now  $(**)$  gives  $24 < 20$ , a contradiction.

Suppose that  $l = 2$ . Notice that  $\gamma < 9$ . Otherwise  $\varepsilon = 0$  and  $t = 2$ , and there are two  $(-2)$ -tips in  $D$ . This gives a contradiction by 1.10 as before.  $(**)$  gives  $4c_2 \tilde{c}_2 < 6(c_2 + \tilde{c}_2)$ . Let  $c_2 \leq \tilde{c}_2$ .

Suppose that  $c_2 = 2$ . We obtain that  $8\tilde{c}_2 < 6(2 + \tilde{c}_2)$ , i.e.,  $\tilde{c}_2 < 6$ . It follows by 3.7.1 that  $\varepsilon > 0$ , so  $\gamma \leq 7$ . Now  $(**)$  gives  $8\tilde{c}_2 < 5(2 + \tilde{c}_2)$ , i.e.,  $\tilde{c}_2 \leq 3$ . If  $\tilde{c}_2 = 2$ , then  $\gamma$  is even by 3.2(a), so  $\gamma \leq 6$  and  $(**)$  gives a contradiction. So  $\tilde{c}_2 = 3$ . From  $(**)$  we obtain  $r = \tilde{r} = 0$ . We have  $P' = 1, \tilde{P}' = \tilde{p}_{\tilde{r}+2}$ . Now  $(*)$  gives  $16 < \tilde{P}'$ , a contradiction since  $\tilde{P}' = 1$  or  $2$ .

Suppose that  $c_2 \geq 3$ . Since  $\gamma \leq 8$ ,  $(**)$  gives  $c_2(4\tilde{c}_2 - 6) < 6\tilde{c}_2$  and  $3(4\tilde{c}_2 - 6) < 6\tilde{c}_2$ . We get  $\tilde{c}_2 < 3$ , a contradiction.

Suppose  $l = 1$ . Then by 3.5.1  $r = \tilde{r} = 0$ , i.e  $c_2 > c_3$  and  $\tilde{c}_2 > \tilde{c}_3$ . We have  $d = 2c_2 + 2\tilde{c}_2$  and the formulas 3.2 take the form

$$(1) \quad \gamma + 3c_2 + 3\tilde{c}_2 = \sum_{i \geq 2} p_i + \sum_{i \geq 2} \tilde{p}_i,$$

$$(2) \quad \gamma + 2c_2^2 + 2\tilde{c}_2^2 + 4c_2\tilde{c}_2 = \sum_{i \geq 2} p_i c_i + \sum_{i \geq 2} \tilde{p}_i \tilde{c}_i.$$

We may assume that  $c_2 \geq \tilde{c}_2$ . The branches meet the  $(-1)$ -curve  $T_1$  created by  $\binom{c_1}{p_1}$  in distinct points. Hence, see 3.3,  $h_\Phi = 1 + (h - 1) + \tilde{h} - 1 = h + \tilde{h} - 1$ . Also,  $T_1$  is branching in  $D'$  and  $(L'_\infty)^2 = -1$ . Hence  $\Psi$  contracts only  $L'_\infty$ , and it is a sprouting contraction, that is  $h_\Psi = 1$ . By 3.3,  $h + \tilde{h} = 4 + \varepsilon + \gamma$ . It follows from 2.5 that  $\varepsilon + \gamma \leq 8$  ( $\gamma = 9$  is ruled out as above). Hence  $h + \tilde{h} \leq 12$ . Since  $c_2 > 1$ ,  $h \geq 2$ . Similarly  $\tilde{h} \geq 2$ . Hence  $h, \tilde{h} \leq 10$ .

We write  $c_2 - p_2 = \mu c_3$ ,  $\tilde{c}_2 - \tilde{p}_2 = \tilde{\mu} \tilde{c}_3$ ,  $c_2 = k c_3$ ,  $\tilde{c}_2 = \tilde{k} \tilde{c}_3$ . Note that  $\mu, \tilde{\mu} \geq 1$  and  $k, \tilde{k} \geq 2$  since  $r, \tilde{r} = 0$ . We rewrite (2) in the form

$$(3) \quad \gamma + c_2^2 + \tilde{c}_2^2 + 4c_2\tilde{c}_2 = -\mu c_2 c_3 + \sum_{i \geq 3} p_i c_i - \tilde{\mu} \tilde{c}_2 \tilde{c}_3 + \sum_{i \geq 3} \tilde{p}_i \tilde{c}_i.$$

We get

$$\gamma + 4c_2\tilde{c}_2 \leq c_3^2(h - 2 - \mu k - k^2) + \tilde{c}_3^2(\tilde{h} - 2 - \tilde{\mu} \tilde{k} - \tilde{k}^2),$$

and, since  $c_2 \geq \tilde{c}_2$ ,

$$(4) \quad \gamma \leq c_3^2(h - 2 - \mu k - k^2) + \tilde{c}_3^2(\tilde{h} - 2 - \tilde{\mu} \tilde{k} - 5\tilde{k}^2).$$

We find  $\tilde{h} - 2 - \tilde{\mu} \tilde{k} - 5\tilde{k}^2 \leq \tilde{h} - 24 < 0$  since  $\tilde{h} \leq 10$ . It follows from (4) that

$$(5) \quad h - 2 - \mu k - k^2 > 0.$$

Since  $h \leq 10$  we get  $7 \geq k(\mu + k) \geq (\mu + 1)(2\mu + 1)$ . We obtain  $\mu = 1$  and  $k = 2$  and  $\binom{c_2}{p_2} = c_3 \binom{2}{1}$ . Hence

(\*\*\*)  $D + E$  has at least three tips, two of them  $(-2)$ -tips. Hence  $\varepsilon > 0$ .

( $\binom{c_1}{p_1}$  and  $\binom{c_2}{p_2}$  produce  $(-2)$ -tips,  $\binom{\tilde{c}_2}{\tilde{p}_2}$  a third tip.)

**Claim.**  $\gamma + \varepsilon \leq 7$ .

*Proof.* Suppose otherwise. Then  $\varepsilon \geq 2$  is ruled out by 2.5,  $\varepsilon = 0$  by (\*\*\*) and 1.10. Hence  $\gamma = 7, \varepsilon = 1$ . By 2.5,  $t = 2$ . Suppose that  $h > 2$ . Then  $D + E$  has at least four  $(-2)$ -tips. It follows from 1.10 that there are four tips, and they are maximal twigs of  $D + E$ . Hence  $c_h = \tilde{c}_h = 2$ . But now it follows from (2) that  $\gamma$  is even, a contradiction. Hence  $h = 2$ . This implies that  $c_3 = 1$ , so  $c_2 = 2$ . Since  $c_2 \geq \tilde{c}_2 > 1$  we have  $\tilde{c}_2 = 2$ . We again reach contradiction with (2).  $\square$

Since  $\gamma + \varepsilon \leq 7$ ,  $h + \tilde{h} \leq 11$ . So  $h \leq 9$ . (5) gives  $h > 8$ . Hence  $h = 9$  and  $\tilde{h} = 2$ . Also  $\gamma + \varepsilon = 7$ . From (2) we get

$$\gamma + 6c_3^2 + 2\tilde{c}_2^2 + 4c_2\tilde{c}_2 = \sum_{i=3}^9 p_i c_i + \tilde{p}_2 \tilde{c}_2 \leq 6c_3^2 + p_9 c_9 + \tilde{c}_2^2.$$

It follows that  $p_9 > 1$  since  $c_9 < 4c_2\tilde{c}_2$ . We have  $\varepsilon > 0$  by (\*\*\*). Since  $\gamma \geq 1$  by 1.8,  $\varepsilon \geq 3$  is ruled by 2.5. If  $\varepsilon = 2$ , then  $\gamma = 5$ , so  $t = 2$  by 2.5, but  $p_h = p_9 > 1$  implies  $t \leq 1$ . Hence  $\varepsilon = 1$  and  $\gamma = 6$ . By 2.5  $t \geq 1$ . Since  $p_9 > 1$ ,  $\tilde{p}_h = \tilde{p}_2 = 1$ . We rewrite (2) and (3) as follows.

$$(6) \quad 5 + 6c_3 + 3\tilde{c}_2 = \sum_{i=2}^9 p_i.$$

$$(7) \quad 6 + 8c_3^2 + 2\tilde{c}_2^2 + 8c_3\tilde{c}_2 = \sum_{i=2}^9 p_i c_i + \tilde{c}_2.$$

From this

$$(8) \quad 6 + 6c_3^2 + 2\tilde{c}_2^2 + 8c_3\tilde{c}_2 = \sum_{i=3}^9 p_i c_i + \tilde{c}_2.$$

since  $c_2 = 2c_3$ . Suppose that there exists  $4 \leq j \leq 8$  such that  $c_j < c_3$ . Then  $c_i p_i \leq \frac{c_3^2}{4}$  and  $\sum_{i=3}^9 p_i c_i \leq (j-3)c_3^2 + (10-j)\frac{c_3^2}{4} \leq 6c_3^2$ . Now (8) gives a contradiction. Hence  $c_i = c_3$  for  $i \leq 8$ . Suppose that  $c_8 > p_8$ . We write  $c_8 - p_8 = \nu c_9$ . Then  $\sum_{i=3}^9 p_i c_i \leq 5c_3^2 + p_8 c_8 + p_9 c_9 = 6c_3^2 - \nu c_3 c_9 + p_9 c_9 \leq 6c_3^2$  and again we reach contradiction with (8). Hence  $p_i = c_i$  for  $i \leq 8$  and  $c_9 = c_3$ . From (6) we get

$$5 + 3\tilde{c}_2 = c_3 + p_9.$$

From (8) we get

$$(9) \quad 6 + 2\tilde{c}_2^2 + 8c_3\tilde{c}_2 \leq p_9 c_3 + \tilde{c}_2.$$

Now  $p_9 = 5 + 3\tilde{c}_2 - c_3$  and (9) gives  $6 + 2\tilde{c}_2^2 + 8c_3\tilde{c}_2 \leq (5 + 3\tilde{c}_2 - c_3)c_3 + \tilde{c}_2$ . Hence  $6 + 2\tilde{c}_2^2 + 8c_3\tilde{c}_2 + c_3^2 \leq 5c_3 + 3c_3\tilde{c}_2 + \tilde{c}_2$ , i.e.,

$$6 + 2\tilde{c}_2^2 + 5c_3\tilde{c}_2 + c_3^2 \leq 5c_3 + \tilde{c}_2.$$

It follows that  $6 + c_3^2 < 5c_3$ . This gives  $2 < c_3 < 3$ , a contradiction.

#### 4. SEPARATION OF BRANCHES II: THE BRANCHES SEPARATE ON THE FIRST BLOWING UP

In this section we rule out the last case in the proof of theorem 4.16, that of the branches separating on the first blowing up. We assume that the branch  $\lambda$  is tangent to  $L_\infty$  and  $\tilde{\lambda}$  is not.

**4.1.** Let  $\tilde{r} + 1$  denotes the number of pairs of  $\tilde{\lambda}$  of the form  $(\frac{\tilde{c}_1}{\tilde{c}_1})$ . So  $\tilde{r} \geq 0$ . We change slightly our usual labeling. The pairs of  $\tilde{\lambda}$  we now label:

$$\left(\frac{\tilde{c}_1}{\tilde{c}_1}\right), \dots, \left(\frac{\tilde{c}_1}{\tilde{c}_1}\right), \left(\frac{\tilde{c}_1}{\tilde{p}_1}\right), \dots, \left(\frac{\tilde{c}_h}{\tilde{p}_h}\right)$$

with either  $\tilde{c}_1 = 1$  and  $\tilde{r} = \tilde{h} = 0$ , in which case we put  $\tilde{p}_1 = 1$ , or  $\tilde{c}_1 > \tilde{p}_1$ . Let  $c_1 - p_1 = \alpha c_2$ . We have  $\alpha \geq 2$  since otherwise we may, as in 3.6, pass to an embedding with smaller resolution tree.

Let  $T_1$  (resp.  $\tilde{T}_1$ ) be the proper transform in  $\overline{S}$  of the  $(-1)$ -curve produced by the pair  $(\frac{c_1}{p_1})$  (resp.  $(\frac{\tilde{c}_1}{\tilde{p}_1})$ ). Let  $C$  (resp.  $\tilde{C}$ ) be the  $(-1)$ -curve produced by the last pair in the HN-sequence for  $\lambda$  (resp.  $\tilde{\lambda}$ ). Since  $\alpha \geq 2$  it is clear that  $T_1, \tilde{T}_1$  and  $C, \tilde{C}, E'$  are not touched by  $\Psi$ , so have the same self-intersection in  $\overline{S}'$  and  $\overline{S}$ . In particular  $\gamma' = E'^2 = E^2 = \gamma$ .

Let  $S^\dagger$  be the surface obtained by the first blowup. Let  $H^\dagger, L^\dagger, E^\dagger$  be the proper transforms in  $S^\dagger$  of the tangent line  $H$  to  $\tilde{\lambda}, L_\infty, \overline{U}$ . Then  $H^\dagger, L^\dagger$  are fibers of a  $\mathbb{P}^1$ -ruling of  $S^\dagger$ . We have  $E^\dagger \cdot L^\dagger = \lambda \cdot L^\dagger = c_1 - p_1$  and  $E^\dagger \cdot H^\dagger = \tilde{\lambda} \cdot H^\dagger + f$ , where  $f$  is the intersection of  $H$  and  $\overline{U}$  at finite distance. We have  $f \geq 2$  since otherwise  $H$  is a good asymptote. If  $\tilde{r} = 0$ , then  $\tilde{\lambda} \cdot H^\dagger = \tilde{p}_1$ . If  $\tilde{r} > 0$ , then  $\tilde{\lambda} \cdot H' \geq \tilde{c}_1$ . Hence we have the following.

**Lemma 4.2.** (a)  $c_1 - p_1 \geq \tilde{p}_1 + 2 \geq 3$ .

(b) If  $\tilde{r} > 0$ , then  $c_1 - p_1 \geq \tilde{c}_1 + 2 \geq 4$ .

**4.3.** The formulas 3.2 take form

$$(1) \quad \gamma + 2c_1 + \tilde{c}_1 = \sum_{i \geq 1} p_i + \tilde{r}\tilde{c}_1 + \sum_{i \geq 1} \tilde{p}_i$$

and

$$(2) \quad \gamma + c_1^2 + 2c_1\tilde{c}_1 = \sum_{i \geq 1} p_i c_i + \tilde{r}\tilde{c}_1^2 + \sum_{i \geq 1} \tilde{p}_i \tilde{c}_i + 2p_1\tilde{c}_1.$$

We multiply (1) by  $\tilde{c}_1$  and subtract (2). We obtain

$$(3) \quad \gamma(\tilde{c}_1 - 1) = (c_1 + \tilde{c}_1)(c_1 - \tilde{c}_1 - p_1) + \sum_{i \geq 2} p_i(\tilde{c}_1 - c_i) + \sum_{i \geq 2} \tilde{p}_i(\tilde{c}_1 - \tilde{c}_i).$$

**Lemma 4.4.**  $\gamma \leq 8$ .



*Proof.* Suppose that  $\gamma = 9$ . By 2.5,  $\varepsilon = 0$  and  $t = 2$ . Hence for both  $\lambda$  and  $\tilde{\lambda}$  we have the situation described in 2.2.1, i.e., we have two maximal twigs of  $D + E$  composed of  $(-2)$ -curves. If either of these has more than one component, or if  $D + E$  has a third maximal twig, we reach a contradiction with 1.10. Hence  $D + E$  has precisely two maximal twigs, and they are  $(-2)$ -tips. It follows that  $h = \tilde{h} = 1$ . Let

$$L_\infty - -T - -T_1$$

be the upper chain created by the pair  $\binom{c_1}{p_1}$ , i.e., the chain having  $L_\infty$  and  $T_1$  as tips. Then the chain  $L_\infty - -T$  contracts to a  $(-2)$ -curve. So either

- (i)  $L_\infty^2 = -2$  and  $T = \emptyset$  or
- (ii)  $L_\infty^2 = -1$  and  $T$  has the form  $(-2) - \dots - (-2) - (-3)$  with a number  $l \geq 0$  of  $(-2)$ -curves.

We find  $p_1 = 1, c_1 = 3$  in the first case and  $p_1 = 2l + 3, c_1 = 2l + 5$  in the second and we reach contradiction with 4.2(a).  $\square$

**Lemma 4.5.**  $\tilde{c}_1 > 1$ , i.e.,  $\tilde{\lambda}$  is not smooth. In particular,  $\tilde{h} \geq 1$  and  $\tilde{c}_{\tilde{h}} > \tilde{p}_{\tilde{h}}$ .

*Proof.* Suppose that  $\tilde{c}_1 = 1$ . The formulas 4.3(1) and (2) take the form

$$(1) \quad \gamma + 2c_1 + 1 = p_1 + \sum_{i \geq 2} p_i$$

and

$$(2) \quad \gamma + c_1^2 + 2c_1 = p_1 c_1 + 2p_1 + \sum_{i \geq 2} p_i c_i.$$

We write them in the following form.

$$(3) \quad \gamma + 1 + c_1 + \alpha c_2 = \sum_{i \geq 2} p_i$$

and

$$(4) \quad \gamma + \alpha c_1 c_2 + 2\alpha c_2 = \sum_{i \geq 2} p_i c_i.$$

We multiply (3) by  $c_2$  and subtract (4). We get

$$(5) \quad c_2(1 + \gamma) + c_2 c_1 + \alpha c_2^2 \geq \gamma + \alpha c_1 c_2 + 2\alpha c_2.$$

From this

$$1 + \gamma + c_1 + \alpha c_2 > \alpha c_1 + 2\alpha.$$

Let  $c_1 = kc_2, p_1 = lc_2$ . Then  $\alpha = k - l$ . We get

$$(6) \quad \gamma - 2\alpha \geq c_2(k\alpha - \alpha - k).$$

Suppose that  $\alpha \geq 3$ . Then  $k = \alpha + l \geq 4$ . We obtain  $\gamma - 6 \geq c_2(2k - 3) \geq 5c_2$ , a contradiction since  $\gamma \leq 8$ . Thus  $\alpha = 2$ . From (5) we get

$$c_2^2(k - 2) + c_2(3 - \gamma) + \gamma \geq 0.$$

Therefore  $\Delta = (3 - \gamma)^2 - 4\gamma(k - 2) \leq 0$ . Since  $k \geq \alpha + 1 = 3$  we have  $(3 - \gamma)^2 - 4\gamma \geq 0$  and finally  $\gamma^2 - 10\gamma + 9 \geq 0$ . From this, since  $\gamma > 2\alpha = 4$  by (6), we obtain  $\gamma \geq 9$ , a contradiction in view of 4.4.  $\square$

**Lemma 4.6.**  $\tilde{c}_1 > c_i$  for  $i \geq 2$ .

*Proof.* It is enough to show that  $c_2 \geq \tilde{c}_1$  is not possible. Multiply 4.3(1) by  $c_2$  and subtract 4.3(2). We obtain

$$\gamma(c_2 - 1) = -2c_1c_2 - \tilde{c}_1c_2 + c_1^2 + 2c_1\tilde{c}_1 + p_1c_2 - p_1c_1 - 2p_1\tilde{c}_1 + \sum_{i \geq 2} p_i(c_2 - c_i) + \sum_{i \geq 2} \tilde{p}_i(c_2 - \tilde{c}_i) + \tilde{r}\tilde{c}_1(c_2 - \tilde{c}_1) + \tilde{p}_1(c_2 - \tilde{c}_1).$$

Let  $c_1 = kc_2$ ,  $p_1 = lc_2$ . Then  $\alpha = k - l$ , hence  $k \geq l + 2$ .

If  $c_2 \geq \tilde{c}_1$  we get

$$\gamma(c_2 - 1) > (-2k + k^2 + l - kl)c_2^2 + c_2(-\tilde{c}_1 + 2k\tilde{c}_1 - 2l\tilde{c}_1).$$

From this

$$\gamma > (-2k + k^2 + l - kl)c_2 - \tilde{c}_1 + 2k\tilde{c}_1 - 2l\tilde{c}_1.$$

Now  $-2k + k^2 + l - kl = (k - l)(k - 2) - l \geq 2(k - 2) - l = k + k - l - 4 \geq k - 2 \geq 1$ . Since  $c_2 > \tilde{c}_1$  we obtain

$$\gamma > \tilde{c}_1(k^2 - kl - l - 1) = \tilde{c}_1(k + 1)(k - l - 1) \geq 4\tilde{c}_1.$$

Now  $\gamma \leq 8$  by 4.4, hence  $\tilde{c}_1 < 2$ , a contradiction in view of 4.5

□

**Lemma 4.7.** *Let  $\beta = c_1 - p_1 - \tilde{c}_1$ . If  $\tilde{r} > 0$  then  $2 \leq \beta \leq 3$ .*

*Proof.*  $\tilde{r} > 0$  implies  $\beta \geq 2$  by 4.2(b). By 4.3(3), 4.6 and 4.2 we find  $\gamma(\tilde{c}_1 - 1) \geq \beta(c_1 + \tilde{c}_1) \geq \beta(2\tilde{c}_1 + 3)$ . In view of 4.6 this gives  $\beta < \frac{\gamma}{2} \leq 4$ . □

**4.8.** We consider again the surface  $Y$  introduced in 2.4.1. Let  $Q_1$  (resp.  $\tilde{Q}_1$ ) denote the maximal twig of  $D + E$  which meets  $C$  (resp.  $\tilde{C}$ ). If  $h > 1$  then  $Q_1$  is the lower subchain produced by the pair  $\binom{c_h}{p_h}$ . If  $h = 1$  then  $Q_1$  is the image under  $\Psi$  of the maximal twig of  $D' + E'$  which has  $L'_\infty$  as a tip. In any case  $\tilde{Q}_1$  is the lower subchain produced by the pair  $\binom{\tilde{c}_h}{\tilde{p}_h}$ . We write  $D = Q_1 + C + Q_0 + \tilde{C} + \tilde{Q}_1$  and put

$$\text{4.8.1} \quad Q = Q_1 + Q_0 + \tilde{Q}_1 + E \quad \text{and} \quad Y = \overline{S} \setminus Q.$$

We note

$$\text{4.8.2} \quad \chi(Y) = -1.$$

**Lemma 4.9.** *If  $\gamma \geq 6$  then  $2K_{\overline{S}} + Q \geq 0$ . In particular  $\overline{\kappa}(Y) \geq 0$ .*

*Proof.* If  $\gamma \geq 6$  then  $\varepsilon = 0$  or  $1$  by 2.5. As in 2.4.1 we have  $K_{\overline{S}} \cdot (K_{\overline{S}} + Q) = K_{\overline{S}} \cdot (K_{\overline{S}} + D + E) - K_{\overline{S}} \cdot C - K_{\overline{S}} \cdot \tilde{C} = 4 - \varepsilon$ . If  $\varepsilon = 0$  we obtain the result as in 2.4.1 and 2.4.2.

Suppose that  $\varepsilon = 1$ . We have  $K_{\overline{S}} \cdot (K_{\overline{S}} + Q) = 3$ . By 2.5 we have  $t \geq 1$ . Hence  $Q_1$  or  $\tilde{Q}_1$ , say  $\tilde{Q}_1$ , consists of  $(-2)$ -curves. Then the Riemann-Roch Theorem gives  $h^0(-K_{\overline{S}} - Q_0 - Q_1 - E) + h^0(2K_{\overline{S}} + Q_0 + Q_1 + E) > 0$ . By 2.4.2 we have  $2K_{\overline{S}} + D + E \geq 0$ . If  $-K_{\overline{S}} - Q_0 - Q_1 - E \geq 0$  then  $K_{\overline{S}} + C + \tilde{C} + \tilde{Q}_1 = 2K_{\overline{S}} + D + E + (-K_{\overline{S}} - Q_0 - Q_1 - E) \geq 0$ . This implies that  $K_{\overline{S}} \geq 0$ , a contradiction. Thus  $2K_{\overline{S}} + Q_0 + Q_1 + E \geq 0$  and hence  $2K_{\overline{S}} + Q \geq 0$ . □

**Lemma 4.10.** *If  $\gamma \geq 6$  then the pair  $(\overline{S}, Q)$  is almost minimal.*

*Proof.* Suppose that  $Q_0$  is contractible (to a quotient singular point), i.e., has negative definite intersection matrix and is a chain or a contractible fork. Then  $Q$  has negative definite intersection matrix and the result follows as in 2.4.3.

Suppose that  $Q_0$  is not contractible and that  $(\overline{S}, Q)$  is not almost minimal. We need the following.

**Sublemma** *There is no  $(-1)$ -curve  $L$  in  $\overline{S}$  such that  $L \cdot Q_0 = 0$ ,  $L$  meets two connected components of  $Q_1 + E + \tilde{Q}_1$  and together with these components contracts to a smooth point.*

*Proof.* Suppose that such an  $L$  exists. Let  $\pi: \bar{S} \rightarrow \bar{X}$  be the contraction of  $L$  and the precisely two connected components of  $Q_1 + E + \tilde{Q}_1$  it meets to a smooth point  $q_1$ . Let  $Q_2$  be the third connected component. The surface  $\bar{S} \setminus Q_0$  is simply connected since it contains  $\mathbb{C}^2$ . Therefore  $X = \bar{X} \setminus Q_0$  is simply connected. Let  $\bar{X} \rightarrow X'$  be the contraction of  $Q_2$  to a cyclic singular point  $q_2$ . Then  $X' = \bar{X}' \setminus Q_0$  is simply connected. It is also easy to compute that  $b_2(X') = 0$ . Hence  $X'$  is contractible. Moreover  $\bar{\kappa}(X') = \bar{\kappa}(X) = \bar{\kappa}(\bar{S} \setminus Q_0) = -\infty$ . By [KR2] the logarithmic Kodaira dimension of the smooth locus of  $X'$  is negative. Since  $q_1$  is smooth,  $\bar{\kappa}(X' \setminus \{q_1, q_2\}) = \bar{\kappa}(\bar{S} \setminus (Q \cup L)) = -\infty$ . It follows that  $\bar{\kappa}(Y) = \bar{\kappa}(S \setminus (Q_1 + E + \tilde{Q}_1)) = -\infty$ , a contradiction in view of 4.9.  $\square$

Let  $(\bar{Y}', T')$  be an almost minimal model of  $(\bar{Y}, Q)$ .  $\bar{Y}'$  is obtained from  $\bar{S}$  by a sequence of birational morphisms  $p_i: \bar{Y}_i \rightarrow \bar{Y}_{i+1}$ ,  $\bar{S} = \bar{Y}_0 \rightarrow \bar{Y}_1 \rightarrow \cdots \rightarrow \bar{Y}_\ell = \bar{Y}'$ . Let  $T_i = (p_{i-1})_*(T_{i-1})$ ,  $T_0 = Q$ ,  $T' = T_\ell$ . Let  $Y_i = \bar{Y}_i \setminus T_i$ . For every  $i$  there exists a  $(-1)$ -curve  $C_i \not\subset T_i$  such that  $p_i: \bar{Y}_i \rightarrow \bar{Y}_{i+1}$  is the *NC-minimalization* of  $C_i + T_i$ . Finally, for the almost minimal model  $(\bar{Y}', T')$ , the negative part  $(K_{\bar{Y}'} + T')^-$  coincides with the bark  $\text{Bk}(T')$ . The contractions in this process involve only curves (or their images) contained in the support of  $(K_{\bar{Y}} + T_0)^-$ . We put

$$e(\bar{Y}_i, T_i) = \chi(\bar{Y}_i \setminus T_i) + \#\{\text{connected components of } T_i\}.$$

We find by an elementary calculation that  $e(\bar{Y}_{i+1}, T_{i+1}) = e(\bar{Y}_i, T_i) - 1$ . Hence  $e(\bar{Y}', T') = e(\bar{S}, Q) - \ell = 3 - \ell$ .

Let  $k$  denote the number of connected components of  $T'$  which contract to quotient singularities, with local fundamental groups  $G_j$ . Let  $u$  denotes the number of connected components of  $T'$ . By 1.13 we have

$$\chi(Y') + \frac{k}{2} \geq \chi(Y') + \sum_{i=1}^k \frac{1}{|G_j|} \geq 0.$$

We have  $\chi(Y') = e(\bar{Y}', T') - u = 3 - \ell - u$ . We obtain

$$3 - \ell - u + \frac{k}{2} \geq 0.$$

Since  $Q_0$  is not contractible,  $k \leq u - 1$ . Also  $\ell \geq 1$  since  $(\bar{S}, Q)$  is not almost minimal. We obtain  $u \leq 3$  and  $k \leq 2$ . By the Sublemma above,  $\chi(Y') \leq \chi(Y) = -1$  (in the minimalization process  $\chi(Y_{i+1}) > \chi(Y_i)$  if and only if  $C_i$  meets two connected components of  $T_i$  and contracts to a smooth point together with these connected components). From 1.13 we get that  $k > 1$ . Hence  $k = 2$ ,  $\ell = 1$ ,  $u = 3$ . Also  $\chi(Y') = -1 = \chi(Y)$ . Again by 1.13

(\*) the two contractible connected components of  $T'$  are  $(-2)$ -curves.

We claim that  $C_0$  meets  $Q_0$ .

Suppose otherwise. Suppose that  $C_0$  meets only one connected component  $Q_2$  of  $Q_1 + E + \tilde{Q}_1$ . Now  $\chi(Y') = \chi(Y)$  implies that  $C_0 + Q_2$  must contract to a smooth point. It follows that  $Q_2 \neq E$  since  $E$  is not a  $(-2)$ -curve (we have  $\gamma \geq 6$ ) and hence  $E + C_0$  cannot contract to a smooth point. Therefore  $E$  is untouched under  $p_0$ , so  $E^2 \neq -2$  in  $T'$ , and we have a contradiction to (\*). Thus  $C_0$  meets two connected components of  $Q_1 + E + \tilde{Q}_1$  and together with these components contracts to a  $(-2)$ -curve. Let  $\bar{X}$  be the image of  $\bar{Y}'$  under the contraction of the two connected components of  $T'$  that are  $(-2)$ -curves to singular points. Put  $X = \bar{X} \setminus Q_0$ . We have  $\bar{\kappa}(X) = -\infty$ ,  $X$  is simply-connected and has trivial Betti numbers. Hence  $X$  is contractible. By [KR2] the smooth locus of  $X$  has negative Kodaira dimension. It follows that  $\bar{\kappa}(Y) = -\infty$ , in contradiction to 4.9.

Hence  $C_0$  meets  $Q_0$  and, since  $\chi(Y') = \chi(Y)$ , one of connected components of  $Q_1 + E + \tilde{Q}_1$ . The other two connected components are  $(-2)$ -curves. It follows that  $C_0$  meets  $E$  and that  $Q_1, \tilde{Q}_1$  are  $(-2)$ -curves.

Suppose that  $h > 1$ . Then  $d(Q_1) = c_h = 2$ ,  $d(\tilde{Q}_1) = \tilde{c}_h = 2$ . By 4.3(2), 4 divides  $\gamma$ . Thus  $\gamma = 8$ ,  $\varepsilon = 0$ . Now we get contradiction with 1.10 since  $D + E$  has two  $(-2)$ -tips and at least one

other maximal twig which meets  $T_1$ . Hence  $h = 1$  and  $Q_1$  is a tip of  $D + E$  which meets  $T_1$ . We reach a contradiction as in the proof of 4.4.  $\square$

**4.11.** Put  $\omega = h_\Psi$ . We have  $h_\Phi = 1 + \tilde{r} + \tilde{h} + h - 1$ . By 3.3 we obtain

$$\tilde{r} + h + \tilde{h} = 2 + \varepsilon + \gamma + \omega.$$

**Lemma 4.12.** *If  $\gamma \geq 5$  then  $Q_0$  is not a chain.*

*Proof.* We put  $P = \sum_{i \geq 2} p_i$ ,  $\tilde{P} = \sum_{i \geq 2} \tilde{p}_i$ . With  $\beta$  as in 4.7 we get from 4.3

$$(*) \quad \gamma(\tilde{c}_1 - 1) = \beta(c_1 + \tilde{c}_1) + \sum_{i \geq 2} p_i(\tilde{c}_1 - c_i) + \sum_{i \geq 2} \tilde{p}_i(\tilde{c}_1 - \tilde{c}_i).$$

Suppose that  $Q_0$  is a chain. We then have four cases:

- (a)  $h = 1, \tilde{h} = 1$   
or
- (b)  $h = 1, \tilde{h} = 2, \tilde{p}_2 = 1$   
or
- (c)  $h = 2, p_2 = 1, \tilde{h} = 1$   
or
- (d)  $h = 2, p_2 = 1, \tilde{h} = 2, \tilde{p}_2 = 1$ .

We note the following.

- (i) If  $h = 2$ , then  $\lambda$  produces two tips in  $D + E$ , one of them a  $(-2)$ -tip.
- (ii) If  $\tilde{p}_{\tilde{h}} = 1$ , in particular if  $\tilde{h} = 2$ , then  $\tilde{\lambda}$  produces a  $(-2)$ -tip in  $D + E$ .

We observe that  $h + \tilde{h} \leq 4$ . From 4.11 we get  $\tilde{r} \geq 3$ . By 4.7 we have  $2 \leq \beta \leq 3$ . Notice that  $P + \tilde{P} = h + \tilde{h} - 2$ . From 4.11 we get  $P + \tilde{P} \geq \gamma - \tilde{r}$ . We have  $c_1 - p_1 = \tilde{c}_1 + \beta$ . From 4.3(1) we get

$$\gamma + c_1 + \beta + 2\tilde{c}_1 \geq \tilde{p}_1 + \tilde{r}(\tilde{c}_1 - 1) + \gamma.$$

So

$$c_1 + \tilde{c}_1 \geq \tilde{p}_1 - \beta + \tilde{r}(\tilde{c}_1 - 1) - \tilde{c}_1.$$

From (\*) we obtain

$$(**) \quad \gamma\tilde{c}_1 - \gamma \geq \beta(\tilde{p}_1 - \beta + \tilde{r}(\tilde{c}_1 - 1) - \tilde{c}_1) + \sum_{i \geq 2} p_i(\tilde{c}_1 - c_i) + \sum_{i \geq 2} \tilde{p}_i(\tilde{c}_1 - \tilde{c}_i).$$

- (1) Suppose that  $\beta = 3$ . From 4.11 we have  $\tilde{r} \geq \gamma - 2$  since  $h + \tilde{h} \leq 4$ . Using this we get

$$2\gamma + 3 \geq \tilde{c}_1(2\gamma - 9) + 3\tilde{p}_1.$$

Since  $\beta = 3$ ,  $\gamma \geq 7$  by (\*). We obtain  $17 \geq 5\tilde{c}_1 + 3\tilde{p}_1$ . This implies  $\tilde{c}_1 = 2$ ,  $\tilde{p}_1 = 1$ ,  $\tilde{h} = 1$ . From (\*\*) we now obtain  $\tilde{r} \leq \gamma - 1$ . By 4.11,  $1 + h + \gamma - 1 \geq 2 + \varepsilon + \gamma$  hence  $h \geq 2 + \varepsilon$ . This gives  $\varepsilon = 0$ ,  $h = 2$ . In view of (i) and (ii) we reach contradiction with 1.10.

- (2) Suppose that  $\beta = 2$ .

(2.1) Suppose also that  $\tilde{r} \geq \gamma - 1$ . (\*\*) gives

$$(***) \quad \gamma + 2 \geq 2\tilde{p}_1 + (\gamma - 4)\tilde{c}_1.$$

Since  $\gamma \geq 5$  we get  $7 \geq 2\tilde{p}_1 + \tilde{c}_1$ . This implies  $\tilde{p}_1 = 1$  or  $\tilde{p}_1 = 2$  and  $\tilde{c}_1 = 3$ . In both cases  $\tilde{h} = 1$ .

(2.1.1) Suppose also  $h = 2$  and  $\tilde{p}_1 = 1$ . Then  $\varepsilon = 1$ , otherwise we reach contradiction with 1.10 as above. So  $\tilde{r} \geq \gamma$  by 4.11 and (\*\*) gives  $\gamma + 4 \geq (\gamma - 2)\tilde{c}_1 + 2\tilde{p}_1 + p_2(\tilde{c}_1 - c_2)$ . For  $\gamma \geq 6$  we get  $10 \geq 2\tilde{p}_1 + 4\tilde{c}_1 + p_2(\tilde{c}_1 - c_2)$ , so  $5 > \tilde{p}_1 + 2\tilde{c}_1$  in view of 4.6 and we have a contradiction since  $\tilde{c}_1 \geq 2$ . For  $\gamma = 5$  we get  $\tilde{c}_1 = 2$ ,  $\tilde{c}_1 - c_2 = 1$  and hence  $c_2 = 1$ . But then  $h = 1$ .

(2.1.2) Suppose also  $h = 2$  and  $\tilde{p}_1 = 2$ . Then  $\gamma = 5$  by (\*\*\*). Now (\*\*) gives  $15 \geq 4\tilde{r}$ , but  $\tilde{r} \geq \gamma - 1 = 4$ , a contradiction.

(2.1.3) Suppose also  $h = 1$ . If  $\varepsilon + \omega \geq 1$  then 4.11 gives  $\tilde{r} \geq \gamma + 1$  and  $(**)$  gives  $\gamma + 6 \geq 2\tilde{p}_1 + \gamma\tilde{c}_1$  and further  $11 \geq 2\tilde{p}_1 + 5\tilde{c}_1$ ; a contradiction. Hence  $\varepsilon = \omega = 0$  and  $\tilde{r} = \gamma$ . From 4.3(1) and (3) we get

$$\gamma + 2c_1 + \tilde{c}_1 = p_1 + \tilde{p}_1 + \gamma\tilde{c}_1.$$

and

$$\gamma(\tilde{c}_1 - 1) = 2(c_1 + \tilde{c}_1).$$

From the second equality we have  $\gamma\tilde{c}_1 = \gamma + 2c_1 + 2\tilde{c}_1$ . We substitute it to the first equality and get

$$\gamma = p_1 + \tilde{p}_1 + \tilde{c}_1 + \gamma,$$

a contradiction.

(2.2) Suppose also that  $\tilde{r} \leq \gamma - 2$ . From 4.11 we obtain  $\gamma - 2 + h + \tilde{h} \geq 2 + \varepsilon + \gamma + \varepsilon$ , i.e.,  $h + \tilde{h} \geq 4 + \varepsilon + \omega$ . It gives  $h = \tilde{h} = 2$  and  $\varepsilon = 0$ . We reach contradiction with 1.10 as before.  $\square$

**Lemma 4.13.** *If  $\gamma \geq 6$  then  $Q_0$  is not a contractible fork.*

*Proof.* Let  $H'$  the exceptional curve produced by the first blowing up in  $\Phi^{-1}$ . Let  $H$  denotes the proper transform of  $H'$  in  $\bar{S}$ . In view of 4.5 and  $c_1 > p_1$  we have to blow up at least twice on  $H'$ . Hence  $H^2 \leq -3$ .

Suppose that  $Q_0$  is a fork. Then either  $T_1$  or  $\tilde{T}_1$  is a branching component in  $Q_0$ .

Suppose  $T_1$  is branching. Then the branches are:  $R_1$ , containing  $\Psi(L'_\infty)$ ;  $R_2$ , containing  $H$  and  $\tilde{T}_1$ ;  $R_3$ , meeting  $C$ .  $R_1$  and  $R_2$  are maximal twigs of  $D + E$ .

Suppose  $\tilde{T}_1$  is branching. Then the branches are:  $R_1$ , containing  $\Psi(L'_\infty), T_1, H$ ;  $R_2$ , the lower part of the chain produced by  $(\frac{\tilde{c}_1}{\tilde{p}_1})$ ;  $R_3$ , meeting  $\tilde{C}$ .  $R_1$  and  $R_3$  are maximal twigs of  $D + E$ .

Suppose that  $Q_0$  is a contractible fork, but not of type  $(2, 2, n)$ . Suppose that  $T_1$  is a branching component in  $Q_0$ .

If  $h > 2$ , then  $R_3$  has a  $(\leq -3)$ -component and hence at most two components. It follows that  $h \leq 3$ . Also  $\tilde{h} = 1$  or  $\tilde{h} = 2$  and  $\tilde{p}_2 = 1$ . In any case  $h + \tilde{h} \leq 5$ . By 4.11,  $\tilde{r} \geq 3$ . But now the twig  $R_2$  has at least 4 components and one of them,  $H$ , is a  $(\leq -3)$ -curve. This is impossible.

Suppose that  $\tilde{T}_1$  is a branching component. Again  $h + \tilde{h} \leq 5$ , so  $\tilde{r} \geq 3$ . It follows that  $R_1$  contains at least 4 components and we reach a contradiction as above.

Suppose that  $Q_0$  is contractible of type  $(2, 2, n)$ .

Suppose that  $T_1$  is a branching component in  $Q_0$ . Since  $H^2 \leq -3$ ,  $R_2$  is the "long"  $n$ -twig of  $Q_0$  and  $R_1, R_3$  are single  $(-2)$ -curves. We have  $d(Q_0) = 4(n(b-1) - \tilde{n})$  where  $\tilde{n}$  denotes the determinant of the twig  $R_2$  with the tip of  $R_2$  meeting  $T_1$  removed and  $b = -T_1^2$ . We have  $h = 2$  and  $p_2 = 2$  since  $R_3$  is a single  $(-2)$ -curve. So  $c_2 > p_2$ , which implies in particular that  $b \geq 3$ . Since  $R_2$  does not consist of  $(-2)$ -curves we have  $n - \tilde{n} > 1$ . We obtain  $d(Q_0) \geq 4(2n - \tilde{n}) = 4(n + n - \tilde{n}) \geq 4(3 + 2) = 20$ . We have  $d(Q_1) = c_2 \geq 3$ . From 1.13 we get

$$(*) \quad 1 \leq \frac{1}{d(\tilde{Q}_1)} + \frac{1}{d(Q_1)} + \frac{1}{\gamma} + \frac{1}{d(Q_0)} \leq \frac{1}{d(\tilde{Q}_1)} + \frac{1}{3} + \frac{1}{6} + \frac{1}{20}.$$

This implies  $d(\tilde{Q}_1) = 2$ . It follows that  $D + E$  has two  $(-2)$ -tips. Since it has at least three tips,  $\varepsilon = 1$  in view of 1.10. Thus  $\gamma = 6$  or  $7$ .  $(*)$  gives  $1 \leq \frac{1}{2} + \frac{1}{d(Q_1)} + \frac{1}{6} + \frac{1}{20}$ , which implies  $d(Q_1) \leq 3$ . Since  $d(Q_1) = c_2 \geq 3$  we get  $c_2 = 3$ . Moreover, since  $\tilde{c}_1 > c_2$  by 4.6,  $\tilde{h} = 2$ .

Suppose that  $\omega = 0$ . Then  $R_2 = \Psi(L'_\infty)$  and  $\frac{c_1}{c_2} = 3$ ,  $\frac{p_1}{c_2} = 1$ . Hence  $c_1 = 9$ ,  $p_1 = 3$ . From 4.11 we obtain  $\tilde{r} = \gamma - 1$ . Now 4.3(1) gives  $\gamma + 12 = \tilde{p}_1 + (\gamma - 2)\tilde{c}_1$ . Since  $\tilde{h} = 2$ ,  $\tilde{c}_2 \geq 4$  and  $\tilde{p}_1 \geq 2$ . We obtain  $\gamma + 12 \geq 2 + 4(\gamma - 2)$ . This implies  $\gamma \leq 6$ , so  $\gamma = 6$ . Also  $\tilde{c}_1 = 4$  and  $\tilde{p}_1 = 2$ ,  $\tilde{r} = 5$ . From 4.3(3) we obtain  $6(4 - 1) = 13 \cdot 2 + 2(\tilde{c}_1 - 3) + \tilde{c}_1 - 2$ , a contradiction.

Thus  $\omega \geq 1$ . From 4.11 we now get  $\tilde{r} \geq \gamma$ . We have  $c_1 - p_1 = \tilde{c}_1 + \beta$ , so 4.3(1) gives  $\gamma + c_1 + \beta + 2\tilde{c}_1 \geq \tilde{p}_1 + \gamma\tilde{c}_1 + 2 + 1$ , i.e.,  $c_1 \geq \tilde{p}_1 + (\gamma - 2)\tilde{c}_1 + 3 - \gamma - \beta$ . Now 4.3(3) gives

$$\gamma\tilde{c}_1 - \gamma \geq (\tilde{p}_1 + (\gamma - 1)\tilde{c}_1 + 3 - \gamma - \beta)\beta + \tilde{c}_1 - 2 + 2(\tilde{c}_1 - 3).$$

If  $\beta = 3$  then  $2\gamma + 8 \geq 3\tilde{p}_1 + 2\gamma\tilde{c}_1$ . If  $\beta = 2$ , then  $\gamma + 6 \geq 2\tilde{p}_1 + (\gamma + 1)\tilde{c}_1$ . In both cases we get contradiction since  $\gamma = 6$  or  $7$  and  $\tilde{c}_1 \geq 4$ .

Assume that  $\tilde{T}_1$  is a branching in  $Q_0$ . Now  $R_2$  and  $R_3$  are single  $(-2)$ -curves. Hence  $\tilde{h} = 2$ ,  $\frac{\tilde{c}_1}{\tilde{c}_2} = 2$  and  $\tilde{p}_2 = 2$ . We again have  $d(Q_0) \geq 20$  and, by 1.13, we get  $d(Q_1) = 2$  and  $d(\tilde{Q}_1) = \tilde{c}_2 = 3$ . Hence  $\tilde{c}_1 = 6, \tilde{p}_1 = 3$ . From 4.3(3) we get

$$5\gamma = (c_1 + 6)\beta + \tilde{p}_2(\tilde{c}_1 - \tilde{c}_2) + p_2(\tilde{c}_1 - c_2).$$

Suppose that  $h = 1$ . Then  $5\gamma = (c_1 + 6)\beta + 6$ . As above,  $\gamma = 6$  or  $7$ . Since  $\beta = 2$  or  $3$ ,  $\beta$  divides  $\gamma$ . Hence  $\gamma = 6$ . Now  $30 \geq 2c_1 + 18$ , which gives  $c_1 \leq 6 = \tilde{c}_1$ . But  $c_1 = p + 1 + \tilde{c}_1 + \beta > \tilde{c}_1$ . We reach a contradiction.

Suppose that  $h = 2$ . Then  $c_2 = d(Q_1) = 2$  and  $p_2 = 1$ . We get  $5\gamma = (c_1 + 6)\beta + 2(\tilde{c}_1 - \tilde{c}_2) + \tilde{c}_1 - 2 = (c_1 + 6)\beta + 10$ . Since  $\gamma \leq 7$ , we have  $35 \geq 2c_1 + 12 + 10$  and again  $c_1 \leq 6$ , a contradiction.  $\square$

**Proposition 4.14.**  $\gamma \leq 5$ .

*Proof.* Suppose that  $\gamma \geq 6$ . By 4.12 and 4.13,  $Q_0$  is not contractible. By 1.13 we have  $\frac{1}{d(Q_1)} + \frac{1}{\gamma} \geq 1$ . Since  $\gamma \geq 6$  we have

(i)  $d(Q_1) = d(\tilde{Q}_1) = 2$   
or

(ii)  $\{d(Q_1), d(\tilde{Q}_1)\} = \{2, 3\}$ ,  $\gamma = 6$ . In this case  $\bar{\kappa}(Y) = 0$  or  $1$  since otherwise  $((K_{\bar{X}} + D)^+)^2 > 0$  in 1.13. We record that  $(BkE)^2 = -\frac{4}{\gamma}$ . Put  $B_0 = (\text{Bk } Q_1)^2 + (\text{Bk } \tilde{Q}_1)^2$ . Then  $B_0 = -4$  if  $Q, \tilde{Q}$  consist of  $(-2)$ -curves. Otherwise they are single curves and  $B_0 = -\frac{4}{3} - \frac{2}{2} = -\frac{10}{3}$ .

Consider (i). Then  $\tilde{c}_h = d(\tilde{Q}_1) = 2$ . Suppose that  $h > 1$ . Then  $c_h = d(Q_1) = 2$ . By 4.3(2),  $4$  divides  $\gamma$ . Hence  $\gamma = 8$ . So  $\varepsilon = 0$  by 2.4 and we reach contradiction with 1.10. Suppose that  $h = 1$ . Then  $C = T_1$  and  $Q_1$  contains  $\Psi(L'_\infty)$ . We come to contradiction as in the proof of 4.8.

Consider (ii). By 4.9,  $2K_{\bar{S}} + Q \geq 0$ . Let  $K_{\bar{S}} + Q = P + \text{Bk } Q$  be the Zariski decomposition. We have  $P \cdot (K_{\bar{S}} + Q) = P^2 = 0$  since  $\bar{\kappa}(Y) = 0$  or  $1$ . Recall that  $P$  is nef. We get

$$0 = P \cdot (2K_{\bar{S}} + 2Q) = P \cdot (2K_{\bar{S}} + Q) + P \cdot Q \geq P \cdot Q.$$

Hence  $P \cdot Q = P \cdot Q_0 = 0$ . Fujita [Fu1] classifies connected components  $Q_0$  of a boundary divisor of an almost minimal surface such that  $P \cdot Q_0 = 0$ . In our case  $Q_0$  is one of the following:

- (a) a chain,
- (b) a tree with exactly two branching components and four maximal twigs being  $(-2)$ -tips,
- (c) a fork of type  $(d_1, d_2, d_3)$  where  $\frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_3} = 1$ .

Case (a) is ruled out by 4.12.

Consider (b). Then  $(\text{Bk } Q_0)^2 = -2$ . Now  $-4 - \varepsilon = (K_{\bar{S}} + Q)^2 = (\text{Bk } Q)^2 = -2 - \frac{4}{\gamma} + B_0$  and  $\frac{4}{\gamma} + B_0$  is an integer. Since  $\gamma \geq 6$  this implies  $B_0 = -\frac{10}{3}$  and  $\gamma = 6$ . We get  $-4 - \varepsilon = (\text{Bk } Q)^2 = -2 - 4 = -6$ , which gives  $\varepsilon = 2$ , a contradiction by 2.5.

Consider (c). We have

$$(*) \quad -4 - \varepsilon = (K_{\bar{S}} + Q)^2 = (\text{Bk } Q_0)^2 + B_0 - \frac{4}{6}.$$

$Q_0$  is of the type  $(3,3,3)$ ,  $(2,4,4)$  or  $(2,3,6)$ . We find that  $(\text{Bk } Q_0)^2 \leq -1$ . Since  $\varepsilon \leq 1$  it follows from  $(*)$  that  $\varepsilon = 1$ . It follows next from  $(*)$  that  $B_0 = -\frac{10}{3}$ , i.e., that  $Q_1$  and  $\tilde{Q}_1$  are single curves, and that  $(\text{Bk } Q_0)^2 = -1$ . By examining all possibilities we see that every twig of  $Q_0$  is a tip, i.e.,  $\#Q_0 = 4$ . Hence  $\#Q = 7$ ,  $b_2(\bar{S}) = 8$ ,  $K_{\bar{S}}^2 = 2$ . From  $K_{\bar{S}} \cdot (K_{\bar{S}} + Q) = 5$  we get  $K_{\bar{S}} \cdot Q = -7$ . Let  $B$  be the branching component of  $Q_0$ . Examining all possibilities we find that  $B^2 > 0$ . But

$B = T_1$  or  $B = \tilde{T}_1$  and both  $T_1$  and  $\tilde{T}_1$  are untouched under  $\Psi$  and hence are negative curves. We reach a contradiction.  $\square$

**Lemma 4.15.**  $\gamma - \tilde{c}_1 - p_1 - \tilde{p}_1 > 0$ .

*Proof.* Suppose the opposite. By 4.14,  $\gamma \leq 5$ , so  $4 \geq \tilde{c}_1 + p_1 + \tilde{p}_1$ . In view of 4.5 we get  $\tilde{c}_1 = 2, \tilde{p}_1 = 1$  and  $p_1 = 1$ . It follows that  $\tilde{h} = 1$  and  $c_2 = 1$ . Hence  $h = 1$ . By 4.11,  $\tilde{r} \geq \gamma + \varepsilon + \omega$ . Hence  $\tilde{r} > 0$ , otherwise  $\gamma = 0$ , which is impossible by 1.8. By 4.2,  $2 \leq \beta \leq 3$ . 4.3(3) gives  $5 \geq \gamma = (c_1 + 2)\beta$ . It follows that  $c_1 = 0$ , a contradiction.  $\square$

**Theorem 4.16.** *If  $U$  has no good asymptote then the branches of  $\overline{U}$  at infinity can be separated by an automorphism of  $\mathbb{C}^2$ .*

*Proof.* Suppose opposite. By results of section 3 we may assume that things are as in 4.1. By 4.14 we have  $\gamma \leq 5$ .

From 4.3(2) we get

$$(1) \quad \gamma + \alpha c_2 c_1 + 2\alpha c_2 \tilde{c}_1 = \sum_{i \geq 2} p_i c_i + \sum_{i \geq 2} \tilde{p}_i \tilde{c}_i + \tilde{r} \tilde{c}_1^2 + \tilde{p}_1 \tilde{c}_1.$$

Since  $\tilde{p}_1 \leq \alpha c_2 - 2$

$$(2) \quad \gamma + \alpha c_2 c_1 + \alpha c_2 \tilde{c}_1 + 2\tilde{c}_1 \leq \sum_{i \geq 2} p_i c_i + \sum_{i \geq 2} \tilde{p}_i \tilde{c}_i + \tilde{r} \tilde{c}_1^2.$$

4.3(1) takes the form

$$(3) \quad \gamma + 2c_1 + \tilde{c}_1 - p_1 - \tilde{p}_1 = \sum_{i \geq 2} p_i + \sum_{i \geq 2} \tilde{p}_i + \tilde{r} \tilde{c}_1$$

and

$$2(c_1 + \tilde{c}_1) + \gamma - \tilde{c}_1 - p_1 - \tilde{p}_1 = \sum_{i \geq 2} p_i + \sum_{i \geq 2} \tilde{p}_i + \tilde{r} \tilde{c}_1.$$

By 4.15,  $\gamma - \tilde{c}_1 - p_1 - \tilde{p}_1 \leq 0$ . Hence  $c_1 + \tilde{c}_1 \geq \frac{1}{2}(\sum_{i \geq 2} p_i + \sum_{i \geq 2} \tilde{p}_i) + \frac{1}{2}\tilde{r} \tilde{c}_1$ . From (2) we get

$$\gamma + \frac{\alpha c_2}{2}(\sum_{i \geq 2} p_i + \sum_{i \geq 2} \tilde{p}_i + \tilde{r} \tilde{c}_1) + 2\tilde{c}_1 \leq \sum_{i \geq 2} p_i c_i + \sum_{i \geq 2} \tilde{p}_i \tilde{c}_i + \tilde{r} \tilde{c}_1^2.$$

Suppose that  $\tilde{r} = 0$ . Then

$$\gamma + \frac{\alpha c_2}{2}(\sum_{i \geq 2} p_i + \sum_{i \geq 2} \tilde{p}_i) + 2\tilde{c}_1 \leq \sum_{i \geq 2} p_i c_i + \sum_{i \geq 2} \tilde{p}_i \tilde{c}_i.$$

Since  $\alpha \geq 2$  this implies that  $\frac{\alpha c_2}{2} < \tilde{c}_2$  and further  $c_2 < \tilde{c}_2$ . It follows that  $\tilde{c}_2 = \tilde{p}_1$ , otherwise  $\tilde{c}_2 \leq \frac{\tilde{p}_1}{2} \leq \frac{\alpha c_2 - 2}{2}$ . We rewrite (3) as

$$\gamma + c_1 + \alpha c_2 + \tilde{c}_1 - \tilde{c}_2 = \sum_{i \geq 2} p_i + \sum_{i \geq 2} \tilde{p}_i$$

and (1) as

$$\gamma + 2\alpha c_2 \tilde{c}_1 + \alpha c_1 c_2 - \tilde{c}_2 \tilde{c}_1 = \sum_{i \geq 2} p_i c_i + \sum_{i \geq 2} \tilde{p}_i \tilde{c}_i.$$

Multiply the first equality by  $\tilde{c}_2$  and subtract the second one. We obtain

$$\gamma(\tilde{c}_2 - 1) = (\alpha c_2 - \tilde{c}_2)(c_1 + 2\tilde{c}_1 - \tilde{c}_2) + \sum_{i \geq 2} p_i(\tilde{c}_i - c_i) + \sum_{i \geq 2} \tilde{p}_i(\tilde{c}_2 - \tilde{c}_i).$$

Since  $\tilde{c}_2 > \frac{\alpha c_2}{2} \geq c_2 \geq c_i$  for  $i \geq 2$

$$\gamma(\tilde{c}_2 - 1) \geq (\alpha c_2 - \tilde{c}_2)(c_1 + 2\tilde{c}_1 - \tilde{c}_2)$$



We have  $\tilde{c}_1 \geq 2\tilde{c}_2$ . Since  $c_1 > \tilde{p}_1 = \tilde{c}_2$  and  $\alpha c_2 - \tilde{c}_2 = c_1 - p_1 - \tilde{p}_1 \geq 2$  we obtain  $\gamma(\tilde{c}_2 - 1) \geq 2 \cdot 4\tilde{c}_2$ . It follows that  $\gamma = 9$ , a contradiction.

Thus  $\tilde{r} > 0$ . By 4.7,  $2 \leq \beta \leq 3$ .

Suppose that  $\tilde{c}_1 \geq 2c_2$ . Then  $\tilde{c}_1 - c_i \geq \frac{\tilde{c}_1}{2}$  for every  $i \geq 2$ . Also  $\tilde{c}_1 - \tilde{c}_i \geq \frac{\tilde{c}_1}{2}$  for every  $i \geq 2$ .

From 4.3(3) we obtain

$$(4) \quad \gamma(\tilde{c}_1 - 1) = (c_1 + \tilde{c}_1)\beta + \sum_{i \geq 2} p_i(\tilde{c}_1 - c_i) + \sum_{i \geq 2} \tilde{p}_i(\tilde{c}_1 - \tilde{c}_i) \geq (c_1 + \tilde{c}_1)\beta + \frac{\tilde{c}_1}{2} \left( \sum_{i \geq 2} p_i + \sum_{i \geq 2} \tilde{p}_i \right).$$

It follows that  $\gamma \geq 5$ , i.e.,  $\gamma = 5$ , and further  $5 > 4 + \frac{1}{2} \left( \sum_{i \geq 2} p_i + \sum_{i \geq 2} \tilde{p}_i \right)$ . It gives  $\sum_{i \geq 2} p_i + \sum_{i \geq 2} \tilde{p}_i \leq 1$ .

It follows that  $h = 1$  or  $h = 2$  and  $p_2 = 1$ , and similarly for  $\tilde{h}$ ,  $\tilde{p}_1$ . It follows that  $Q_0$  is a chain in contradiction to 4.12.

Hence  $\tilde{c}_1 < 2c_2$ . (4) and 4.6 give  $5\tilde{c}_1 > 2(c_1 + \tilde{c}_1)$  i.e.  $c_1 < \frac{3}{2}\tilde{c}_1$ . But  $c_1 \geq 3c_2 > \frac{3}{2}\tilde{c}_1$  since  $\alpha \geq 2$ , a contradiction. □

## REFERENCES

- [C-NKR] P.Cassou-Nogues, M.Koras, P.Russell, Closed embeddings of  $\mathbb{C}^*$  in  $\mathbb{C}^2$ , part I, J. Algebra **322**(2009)
- [Fu1] T.Fujita, *On the topology of non-complete surfaces*, J. Fac. Sci. Univ. Tokyo, **29**(1982), 503-566.
- [Fu2] T.Fujita, *On Zariski problem*, Proc.Japan.Acad. **55**(1979), 106-110.
- [GM] R.V. Gurjar and M. Miyanishi, Affine lines on logarithmic  $\mathbb{Q}$ -homology planes, Math. Ann. **294** (1992), 463-482.
- [I] S.Itaka, *On logarithmic Kodaira dimension of algebraic varieties*, Complex Analysis and Algebraic Geometry, Iwanami Shoten, Tokyo, 1977, 175-189.
- [Ka] Y. Kawamata, *On the classification of non-compact algebraic surfaces*, Lecture Notes in Mathematics **732**, Springer (1979).
- [Ko] R. Kobayashi, *Uniformization of complex surfaces*, Adv. Stud. Pure Math., **18**(1990), 313-394.
- [Kor] M.Koras,  *$\mathbb{C}^*$  in  $\mathbb{C}^2$  is birationally equivalent to a line*, Affine Algebraic Geometry: The Russell Festschrift, CRM Proceedings & Lecture Notes, 2011.
- [KR1] M. Koras, P. Russell,  *$\mathbb{C}^*$ -actions on  $\mathbb{C}^3$ : The smooth locus is not of hyperbolic type*, J. Algebraic Geometry, **8**(1999), 603-694.
- [KR2] M.Koras, P.Russell, *Contractible affine surfaces with quotient singularities*, Transformation Groups, **12**, 2007, 293-340.
- [L] A. Langer, Logarithmic orbifold Euler numbers of surfaces with applications, Proc. London Math. Soc. (3) **2003**, no.2, 358-396.
- [KM] M. Kumar and P. Murthy: *Curves with negative self intersection on rational surfaces*, J. Math. Kyoto Univ., 22-4(1983), 767-777
- [M] M.Miyanishi, *Open Algebraic Surfaces*, CRM monograph series, Amer.Math.Soc., 2001.
- [Mi] Y. Miyaoka, *The maximal number of quotient singularities on surfaces with given numerical invariants*, Math. Ann., **268**(1984), 159-171., American Math.Society, Providence, Rhode Island, 2001.
- [Ru1] P.Russell, *Hamburger-Noether expansions and approximate roots of polynomials*, Manuscripta Math., **31**(1980), 25-95.
- [Ru2] P. Russell, *On affine-ruled rational surfaces*, Math. Ann., **255**(1981), 287-302.

M.KORAS: INSTITUTE OF MATHEMATICS, UNIVERSITY OF WARSAW AND PETER RUSSELL: DEPARTMENT OF MATHEMATICS, MCGILL UNIVERSITY, MONTREAL, CANADA.

*E-mail address:* koras@mimuw.edu.pl, russell@math.mcgill.ca